THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

62 COLERAINE ROAD, BLACKHEATH, LONDON, S.E. 3.

LONDON

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Intending members are requested to communicate with one of the Secretaries (G. L. Parsons, Peckwater, Eastcote Road, Pinner, Middlesex; Mrs. E. M. Williams, 155 Holden Road, N.12). The subscription to the Association is 15s per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to Mrs. Williams. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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VOL. XXIII. FEBRUARY, 1939. No. 253

THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Mathematical Association was held at King's College, London, on 2nd and 3rd January, 1939. On Monday, 2nd January, the proceedings opened at 2.15 p.m. with the transaction of business; the President, Mr. W. Hope-Jones, was in the chair.

The Report of the Council for 1938 was adopted.*

The Hon. Treasurer presented a statement of accounts for the year ending 31st October, 1938.

On the nomination of the Council, Mr. W. C. Fletcher, C.B., was elected President for the year 1939. Miss Punnett, who has served the Association as one of the Secretaries since 1912, retired from this office, and was elected a Vice-President. The existing Vice-Presidents, the Treasurer, Librarian, and the Editor of the *Gazette*, and Mr. G. L. Parsons as one of the Secretaries, were re-elected. Mrs. E. M. Williams was elected as one of the Secretaries, in place of Miss Punnett. The Auditor was re-elected. Miss M. J. Griffith retired from the Council and was not eligible for re-election. The following were elected as members of the Council for 1939:

Professor G. B. Jeffery,
Mr. W. J. Langford,
Miss W. M. Lehfeldt,
Mr. C. W. Parkes,
Mr. A. W. Riley,
Miss M. A. Hooke,
Mr. F. J. Swan,
Miss M. E. Bowman,
Mr. A. P. Rollett.

The amendments to the Rules of the Association proposed by Professor E. H. Neville were adopted.

Mr. Hope-Jones then delivered his Presidential Address: *Simplicity and Truthfulness in Arithmetic.*† This was followed by Professor W. L. Bragg's lecture: *The Symmetry of Patterns.*

* See pp. 2-5. † See p. 7.

On Tuesday, 3rd January, the President in the chair, the proceedings opened with Mr. A. Buxton's paper, *The Teaching of Applied Mathematics in Technical Colleges*;* this was followed by Mr. Newton's address on *Greenwich Observatory: some aspects of its work*; and a discussion on *The Second Report on the Teaching of Geometry*,* introduced by Mr. A. W. Siddons, Miss M. A. Hooke (a paper read by Miss W. M. Lehfeldt in Miss Hooke's absence), and Mr. H. Beardwood.

The afternoon meeting opened with Mr. W. L. Ferrar's paper, *Algebra in the Higher School Certificate*,* followed by an address from Lord Stamp: *Education and the Statistical Method in Business (with special reference to Railway Statistics)*.*

A Publishers' Exhibition was open during the two days.

An exhibition of books from the Library of the Association was arranged by the Librarian and was on view during the two days. These books represented the Association's remarkable collection of books on geometrical conics, ranging from the sixteenth century to the present day.

REPORT OF THE COUNCIL FOR THE YEAR 1938.

DURING the year 1938, 160 new members have joined the Association, of whom 40 are junior members. The number of members now on the roll is 1714, of whom 9 are honorary members, 112 are life members, 1493 are ordinary members and 100 are junior members.

The Council regrets to have to report the deaths of the following members of the Association: Miss A. V. Beeching, Miss F. Hewitson, Mr. W. E. Jackes, Miss E. M. Richards and Mr. H. C. Smith.

The Branches.

The reports from the Branches show that they continue to be active and vigorous and that their membership is well maintained. Particulars of their meetings have been given in insets in the *Gazette*. Considerable interest has been shown in the Geometry Report published in May. The Cardiff Branch held a meeting at which Mr. C. O. Tuckey gave a much appreciated address on the Report to an audience of well over one hundred; it consisted not only of members of the Cardiff and South-west Wales Branches, but also of teachers from other parts of South Wales. The Liverpool and Manchester Branches also report well-attended meetings and lively discussions on the same subject. The South-west Wales Branch is contemplating co-operation with the Cardiff Branch on the subject of the syllabuses and papers set in the Central Welsh Board Examinations. The London Branch has had successful and very well-attended meetings, the relations of Mathematics and Logic and those of Mathematics and Music having been among the subjects discussed. The Northern Ireland Branch invited the

* To be published later.

members of the Economic Society of Queen's University, Belfast, to one of their meetings at which a paper on " Economics and Mathematics " was read by the Vice-Chancellor of the University.

The formation of a Sheffield Branch has also been considered at a well-attended preliminary meeting.

The composition of the Branches in terms of members, associates and junior members is as follows : Bristol, 20 members, 23 associates ; Cardiff, 47 members, 39 associates, 1 junior member ; Liverpool, 21, 45 ; London, 213, 64, 1 ; Manchester, 36, 69 ; Midland, 39, 34, 1 ; North-eastern, 44, 14 ; Northern Ireland, 4, 31 ; North Wales, 5, 12, 2 ; Southampton, 10, 33 ; South-west Wales, 12, 28 ; Yorkshire, 48, 67, 1 ; Queensland, 12, 17 ; Sydney, N.S.W., 21, 129 ; Victoria, 11, 26.

The Mathematical Gazette.

Vol. XXII contained about 100 pages more than the average of recent volumes : an unusually large number of lengthy articles, including those by Sir Percy Nunn, Professors Carslaw, Hartree, Hogben and de la Vallée Poussin, accounted for this increase.

Professor Wolff of Düsseldorf has written an appreciation of the Association's *Second Report on the Teaching of Geometry* from the German point of view for Vol. XXIII.

The Teaching Committee.

The full Committee met for the first time in January, 1938, immediately after the Annual Meeting of the Association.

At that meeting several sub-committees were appointed, and there was a general discussion of further work that might be attempted.

The terms of reference of these sub-committees and the progress so far made are outlined below. In the May number of the *Gazette* was inserted an appeal to members inviting co-operation in the work of the sub-committees. This produced two replies.

(1) The Teaching of Children up to the Age of 11.

This sub-committee has discussed the kind of work most suitable for these children. It has received great help from some of H.M. Inspectors.

(2) The Revision of the Report on the Teaching of Mathematics to Evening Technical Students.

Mr. A. W. Geary (Northampton Polytechnic), Mr. H. V. Lowry (Woolwich Polytechnic) and Mr. W. Hartas Jackson, H.M.I., have been co-opted to this sub-committee.

(In the October *Gazette* it issued an appeal for applications of elementary mathematics which might be useful in teaching.)

(3) The Presentation of the Historical Side of Mathematics.

The Secretary is collecting information from those working on this subject in America.

(4) The Teaching of Trigonometry.

This sub-committee began work in the autumn, and decided to restrict the scope of its report to the School Certificate course. It has discussed the introduction of trigonometry.

(5) Examinations.

This sub-committee has under consideration many suggestions for constructive criticism.

The Council has decided to increase the number of persons who may be co-opted to any sub-committee to five.

The Editor of the *Gazette* is dealing with suggestions for two series of articles: on points arising from the paper on "Teaching the Complete Duffer" at the last Annual Meeting of the Association, and on the contact of mathematics with other subjects.

In April an invitation was received from the British Council to submit a list of typical mathematical books for an exhibition in Portugal. This was done, and a most appreciative acknowledgment received.

A note on the new regulations for the Higher Certificate examination of the University of London was printed, at the suggestion of a member, in the October *Gazette*.

The Library.

The use made of the Library has remained at the steady level of the last few years. There have been comparatively few accessions, but several irritating gaps have been filled. Progress in the work of cataloguing and labelling has been satisfactory, but in binding it is only possible to keep abreast of current needs and arrears are not being overtaken.

The Problem Bureau.

The Bureau continues to send out a number of solutions to members, and it is hoped that these are found to be useful. Complete solutions to recent sets of Cambridge scholarship papers are being prepared.

The thanks of the Association are due to all those members who have assisted in the work of the Bureau, especially Dr. E. A. Maxwell, who has helped on many occasions.

General.

The *Second Report on the Teaching of Geometry*, published in April, is the largest Report so far issued by the Association. It has been received with interest, and forms a noteworthy addition to the publications of the Association.

A supplementary list of members is being prepared and will be circulated shortly.

The Committee appointed to consider the possibility of affiliation of existing student societies was in favour of providing such facilities

by the creation of Junior Branches. Pending the passing of Rules instituting this type of Branch, the Council provisionally approved as a Junior Branch a Cambridge Society known as The Archimedeans, whose membership is 220.

The Council appointed a Committee to draft Rules relating to the above affiliation and to certain other points which had been raised. This Committee has completely revised the Rules of the Association and will present them for approval at the Annual General Meeting.

Officers and Council.

The Council offers the thanks of the Association to Mr. Hope-Jones for his services during his Presidency and nominates Mr. W. C. Fletcher, C.B., as the incoming President.

The Council has accepted with great regret the resignation of Miss Punnett from the office of Secretary. Miss Punnett has held this office for twenty-seven years, and during the whole of that time has not only carried out the duties of her office in a very efficient manner but has also rendered many signal services to the Association in general. The Council thanks Miss Punnett for the work she has done, both for the Council itself and for the Association as a whole, and proposes that she be elected a Vice-President of the Association. Mrs. E. M. Williams has been nominated as her successor.

The thanks of the Council are once more to be accorded to Professor Neville for continuing his valued work as Librarian, to Mr. Broadbent for his services as Editor of the *Mathematical Gazette*, and to Mr. Gosset Tanner for the work of the Problem Bureau. The Council also thanks Miss M. J. Griffith and Mr. R. M. Gabriel, the retiring members, for work done in the past, especially in connection with the programmes of annual meetings.

GLEANINGS FAR AND NEAR.

1237. The intoxication of our multitudes is negative. It is not to be negotiated with. But many negatives may make an affirmative, as in arithmetic "many littles make a muckle". It is, above all, in negation of political authority that crowd-consciousness hardly counts. If the crowd does wish to repudiate it, they can do it.—R. J. Sender, *Seven Red Sundays*. [Per Mr. S. F. Trustram.]

1238. At nine and a half, to the amazement of all Germany, he (Karl Witte) entered the University of Leipzig. In 1814, before he had passed his fourteenth birthday, he was granted the degree of Ph.D. for a thesis on the "Conchoid of Nicomedes" a curve of the fourth degree.—Bruce, *Psychology and Parenthood*. [Per Mr. W. J. Langford.]

1239. I wish I could write this history in words of one syllable. But it cannot be done. You cannot write a text-book of geometry without reference to a hypotenuse and triangles and a rectangular parallelopiped (*sic*).—Hendrik van Loon, *The Story of Mankind*, p. 221. [Per Mr. W. J. Langford.]

THE TEACHING COMMITTEE.

In the *Report of the Council for* 1938 members will have read an account of the work of several sub-committees of the Teaching Committee.

At the meeting of the Teaching Committee in January, 1939, it was agreed that information on work in progress should appear at frequent intervals in the *Gazette*, for the benefit of members of the Association.

It is hoped that any members who desire to contribute to the work of any of the sub-committees will do so by writing in the first instance to the Secretary of the sub-committee concerned.

The list of sub-committees and their Secretaries is :

The Teaching of Children up to the Age of 11 :
Mrs. E. M. Williams, Goldsmiths' College, University of London.

The Teaching of Mathematics to Technical Students :
A. Buxton, The Technical College, Cardiff.

The Teaching of Trigonometry :
J. C. Manisty, 13 College Street, Winchester.

The Historical Side of Mathematics :
M. Black, Institute of Education, Senate House, London, W.C. 1.

Examinations :
C. T. Daltry, 99 Maze Hill, London, S.E. 10.

The following information may be given concerning the Examinations Sub-Committee. Its members are : A. Robson (Marlborough), *ex officio* ; C. T. Daltry (The Roan School, Greenwich), *ex officio* ; Dr. W. G. Bickley (Imperial College) ; Dr. Bertha Swirles (Girton College) ; T. A. A. Broadbent (R.N. College, Greenwich), with the following co-opted members : G. L. Parsons (Merchant Taylors'), R. T. Hughes (Harrow), A. C. Heath (St. Paul's), Mrs. E. Shuttleworth (Queen's College, W. 1). For the present Dr. Bickley is Chairman and Mr. Daltry is Secretary.

The Sub-Committee has agreed on the following procedure :

(1) To consider comments from members on examination syllabuses or individual questions.

These should be sent either to Mr. Daltry or to one of the Secretaries of the Association to be forwarded to the Examinations Sub-Committee. Will members please note that comments on scholarship papers have sometimes been sent to Mr. Gosset Tanner, the Secretary of the Problem Bureau. Whilst members will continue as before to obtain solutions from Mr. Gosset Tanner, they should send comments on the standard of difficulty or errors in such papers to the Secretary of the Examinations Sub-Committee.

(2) As part of its annual routine to consider the papers set in mathematical subjects (including additional mathematics and mechanics) at the First School Certificate examination of one of the major examining bodies.

In 1939 the papers set by the University of London will be considered.

Comments from members on questions or syllabuses will be considered if submitted to the Sub-Committee.

(3) As a first step towards unifying existing examination syllabuses the Sub-Committee intends to consider one subject at a time, beginning with algebra. Each member is investigating the syllabus of one examining board, and is considering the interpretation of this syllabus as shown in questions set during the past three years.

At the meeting of the Teaching Committee in January 1939, it was agreed to appoint a Sixth Form Sub-Committee to explore the whole field of sixth form mathematics. Further information will be given when the Sub-Committee has appointed its officers.

SIMPLICITY AND TRUTHFULNESS IN ARITHMETIC.

By W. Hope-Jones.

(*Presidential Address to the Mathematical Association, January*, 1939.)

I am no artist, but I believe my artistic friends when they tell me that among the first requisites for good art are simplicity and truthfulness. Architecture is also outside my range; but I have architectural friends too, and they tell me of the need for simplicity and truthfulness in Architecture. After taking my degree in Mathematics I went for nearly a year to a Theological College; and I very soon found out for myself what were the qualities which a Theological College most urgently requires.

"And the moral of that", said the Duchess, "is 'Be what you would seem to be'—or if you'd like it put more simply, 'Never imagine yourself not to be otherwise than what it might appear to others that what you were or might have been was not otherwise than what you had been would have appeared to them to be otherwise'."

To those of you who deal in more advanced branches of Mathematics than come my way, I commend as a suggestion worth thinking over, "Could your teaching be improved by the use of simpler words without sacrifice of truthfulness?" But I have nothing to say to-day about advanced Mathematics: I believe that a child brought up to love simplicity and truthfulness in Arithmetic (assuming one to exist) will naturally import these same qualities into higher work when the time comes for that.

"Checkability."

Now, to begin with, a great deal of very obscure and disgusting work that is shown up to me is defended by its authors on the ground that "it gets the answer right". Pass over for the time the plain fact that excessively often it gets the answer wrong, and consider the question whether the whole object of Mathematics is to get the answer right. How many ways there are of getting the answer right! Looking it out in the crib is the quickest: another way is to sit next to a brainy boy and keep your dividers well sharpened. But for practical purposes, not only at school but in after-life, it is so often necessary to convince somebody else that your answer is right. If you are my fishmonger or greengrocer, you must obtain my agreement to your multiplication and addition before I will pay your bill. Even an Income-tax Collector presents his actual calculations in a form in which you can easily check them, though he has you on the obscurity of his data, in which simplicity and truthfulness are not the most conspicuous features, and which are supported by an authority which has little in common with the "appeal to reason" which is the guiding star of all you good teachers

of Mathematics. I take it then that in Arithmetic our pupils should learn to get answers right by working which will convince a reasonably intelligent reader that they are right. And as one of the first tests of a well-done piece of work I put forward this : " A well-done piece of work is easier to check by reading it than by doing it again from the beginning." I am sorry that there is no good English word to denote the facility with which written work can be checked by reading it : in the absence of that I shall have to call it " checkability " ; and I put it to you that not only would our pupils receive a better education but also we ourselves would live more tolerable lives if we would insist on this " checkability " test : *a well-done piece of work is easier to check by reading it than by doing it again from the beginning.*

A book which came to my notice lately printed as the complete solution of an arithmetical problem :

$$\begin{array}{r} 7\text{s.} \\ £147\ 12\text{s. } 6\text{d.} + £12\ 12\text{s. } 0\text{d.} + 10\text{s. } 6\text{d.} - £17\ 10\text{s. } 0\text{d.} \\ - £101\ 5\text{s. } 6\text{d.} - £2\ 2\text{s. } 0\text{d.}, \end{array}$$

in which it will be observed that the dead figures outnumber the living by twenty to eight. This was described by the author as " a reasoned form ", but to me it seems to fall somewhat below the standard of checkability which I require in work that I have to look over.

Cancelling.

It was the difficulty of checking work in which figures are crossed out that first led me to call in question the utility and necessity of the practice called " cancelling ". The only defence of it which I hear when I discuss the practice with those who are addicted to it is that " it is impossible to get on without it ". Now this is the same defence that you will get if you make a similar criticism of those who find it necessary to consume a large cocktail before, during, after, between, instead of and independently of every meal : " it is impossible to get on without it." But it is probably the experience of many of you that life without so much refreshment is not completely unliveable. And since I gave up cancelling nearly thirty years ago, I have never felt the need for it. My hair has not fallen out, nor my feet become a wilderness of bunions.

Boys insist that cancelling is necessary for a particular question : and when I show them how easy it is without it, they shift their ground and say, " Ah, but you might have another question just a little different, in which you really might have to cancel." Well, I suppose you *might* have a situation in which you *might* have to shoot your grandmother : yet many of us go on year after year without ever meeting it. So I go on year after year, hearing about the question in which I will have to cancel, yet never quite meeting it. Very occasionally I meet a question, generally one specially designed for the purpose, in which my preference for clean working

costs me an extra five or ten seconds; but that is a price which I willingly pay for the many minutes which I gain every year by being able to check my own work—to which I would like to add the many hours which I would save if somebody in Europe would invent and send me a boy who believes in doing good, clean, checkable working. (And my reason for adding "in Europe" is that such a boy has actually been invented, and later I shall tell you something of him; but unfortunately he is a very long way off, and the cost of importing him in large quantities is prohibitive.)

In fractions like

$$^{44}C_6 = \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6!},$$

a mass of cancelling involves great risk of error, as well as making the work uncheckable. A clear arrangement for this is to keep a score-sheet of the occurrence of the various prime factors involved, thus:

2	3	5	7	11	13	41	43
2	1	1	1	1	1	1	1
1	1						
3							
−1	−1	−1					
−2	−1						
−1							
2	0	0	1	1	1	1	1

$$\begin{aligned} ^{44}C_6 &= 2^2 \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 43 \\ &= 4 \cdot 1001 \cdot 1763 \\ &= 7052 \\ &\quad + \ 7052 \\ &= 7059052 \end{aligned}$$

These criticisms of "cancelling" have been based chiefly on the unpleasantness and obscurity of the working, even if the answer is right; but it ought also to be noticed that cancelling is the most frequent cause of wrong answers, first because figures are liable to be lost in the mud, and worse still, because the natural boy, who at eight years old understands the difference between subtracting and dividing, has so often had the distinction completely obliterated in his mind by the age of thirteen, when he has learnt to talk of both indiscriminately by the name of "cancelling". ("Oh sir, Mr. X. taught me that when it cancels out I ought to put down 1, not 0; and now you say that it ought to be 0.")

The Improper Fractions Superstition.

Now the main burden of all that I have to say to you is this—that boys have not enough ingenuity to invent their own bad methods of doing their working, but that they pick them up from those who have charge of the early stages of their education. By the time they reach a Public School they are mostly irretrievably

ruined already, because their conservatism is such that to change a method taught them at their Preparatory Schools is abhorrent to their nature, especially if it has been taught them by a master known to be a good man because he can imitate a pig or has hit a ball through an unpopular neighbour's window. It is too much to ask that such men should learn or teach methods of making difficult questions easy; but could they somehow be restrained from concentrating the child's entire education upon dodges for making easy questions difficult? Unfortunately these good men are often not members of our Association; but you are, and if I can't appeal to the culprits themselves, you are the missionaries who must go among them with your hymn-books and harmoniums and convert them to the cause of clean and checkable working.

No boy, unaided by the perverted ingenuity of some older and more decayed brain, could ever have invented the Improper Fractions Superstition. The Improper Fractions Superstition is this: that whenever you see a number followed by a fraction, such as $4\frac{1}{8}$, you multiply the denominator by the whole number, add in the numerator, and "put it over" the denominator. "Put it over"—an intolerable form of words: there are few if any that do more harm. Boys arrive at Eton cram-full of rules about putting something over something else: you find the speed by putting the distance over the time, or the time over the distance—it is an even chance which you do, and by taking turn and turn about you can ensure a decent proportion of successes.

But to return to the Improper Fractions Superstition. If you look back into your own past, you will recognise that very early in life you understood "four"—unless you are the kind of philosopher who likes proving that nobody ever understands anything. Later you learnt to understand "an eighth", probably before you understood 3 eighths, and certainly long before you understood 33 eighths. After that $4\frac{1}{8}$ is an easy idea: you can eat four biscuits and an eighth of another: you can walk four miles and an eighth of another: round a circle is three whole diameters and nearly a seventh of another, not a seventh of twenty-two diameters. The natural and untutored boy thinks in terms of wholes and bits in preference to improper fractions. And there the natural and untutored boy is quite right, bless his soul; but this sort of thing can't be allowed to last: so presently he is caught and put through the sacred mysteries of the Improper Fractions Superstition; and by the time he comes to Eton, and the first question I ask him is "Can you *prove* anything about the circumference of a circle?", up goes his hand and he crows in triumph "22 over 7", or some variation on it such as "$22\pi r^2$".

I am not talking now of stupid boys: for many years now I have had the top Division of Fourth Form, and this is how nearly all of them would multiply 2·6 by $3\frac{1}{7}$, correct to three figures:

$2{\cdot}6 \times 3\frac{1}{7} = \frac{13}{5} \times \frac{22}{7} = \frac{286}{35}$

$$
\begin{array}{r}
8{\cdot}171 \\
*=35\,)\overline{286} \\
\underline{280} \\
60 \\
\underline{35} \\
250 \\
\underline{245} \\
50
\end{array}
$$

(the extra figure superstition) $= 8{\cdot}17$

$$
\begin{array}{r}
13 \\
22 \\
\hline
26 \\
\underline{26} \\
286
\end{array}
$$

I will not insult you by supposing that you don't know a better way than this of doing it; but I do want to call your attention to this important fact: the right method is based on the conviction that 3 means three, $\frac{1}{7}$ means a seventh, and $3\frac{1}{7}$ means just what it says, three and then another seventh.

$$
\begin{array}{r}
2{\cdot}6 \\
3\frac{1}{7} \\
\hline
7{\cdot}8 \\
{\cdot}37\ldots \text{ (because 50 is nearer} \\
\hline
8{\cdot}17\ldots \quad \text{to 49 than to 56)}
\end{array}
$$

Now there is no point in jeering at the natural mistakes that boys of thirteen and fourteen make; but that is just what I am not doing. These are not the natural mistakes of the young: they are absolutely unnatural, and no boy would ever think of such artificial methods by himself. There is somebody about who is *teaching* boys this stuff, and will continue to do so unless you can catch him and send him to a reformatory.

All boys know that a fraction remains unchanged by dividing top and bottom by the same number (though some of them won't recognise this wording of the rule, but will ask if you mean "cancelling"); but I find that most new boys are shocked and a little incredulous at the news that a fraction remains the same size if you *multiply* top and bottom by the same number. So if you ask them to simplify $\frac{2\frac{1}{2}}{8\frac{1}{3}}$, instead of multiplying top and bottom by 6,

$$\left(\frac{2\frac{1}{2}}{8\frac{1}{3}} = \frac{15}{50} = \frac{3}{10}\right),$$

practically every boy falls back on the Improper Fractions Superstition; and probably most will cancel into the bargain.

$$\frac{2\frac{1}{2}}{8\frac{1}{3}} = \frac{5}{2} \div \frac{25}{3} = \frac{\overset{1}{\cancel{5}}}{2} \times \frac{3}{\underset{5}{\cancel{25}}} = \frac{3}{10}.$$

* This "=" sign is a particular enemy of mine.

Even such an extreme instance as this is not uncommon :

$$5\tfrac{1}{9} \times 7 = \frac{46}{9} \times \frac{7}{1} = \frac{322}{9} = 35\tfrac{7}{9}.$$

In doing Compound Interest, it is rare to meet a boy, or a book, that will treat $4\frac{1}{4}$ as four and a quarter :

C.I. on £713 at $4\frac{1}{4}\%$

$$\left.\begin{array}{r} 28{\cdot}52 \\ +\ 1{\cdot}7825 \end{array}\right\} I_1$$

$$743{\cdot}3025 \quad P_2.$$

Nearly all of them treat it as $\frac{1}{4}\frac{7}{}$, doing a long-multiplication sum if they don't know their 17 times table :

£713	P_1	713
121·21		17
30·3025	I_1	713
743·3025	P_2	4991
		12121

and some will cross out the 121·21 before they write the 30·3025, presumably on the theory that a number is easier to divide when it has first been slaughtered.

Before leaving the subjects of cancelling and the Improper Fractions Superstition, I would have you notice that they are accomplices and form what in modern politics we call "an axis". It is the Improper Fractions Superstition that provides most of the cannon-fodder for the canceller : to get rid of war, get rid of the causes of war ; and to get rid of the slaughter of innocent figures, as of our innocent fellow-creatures, the great requirement is that we should "live in the virtue of that life and power that takes away the occasion of all cancelling" : it was "wars" in the original, but in Arithmetic we may well interpret that as cancelling, remembering that cancelling is the arithmetical equivalent of war.

If in $4\frac{1}{2}$ minutes a train goes $3\frac{1}{2}$ miles, how far will it go in $58\frac{1}{2}$ minutes?

In $4\frac{1}{2}$ minutes it goes $3\frac{1}{2}$ miles ;

$\therefore$,, 1 ,, ,, $\dfrac{3\frac{1}{2}}{4\frac{1}{2}} = \frac{7}{9}$ miles ;

$\therefore$,, $58\frac{1}{2}$,, ,, $\frac{7}{9}$ of $58\frac{1}{2} = 7 \times 6\frac{1}{2} = 45\frac{1}{2}$ miles.

Here we are following the simple principle, "Take easy simplifications in your stride". But because this method gives nothing to cancel, many boys are taught to do this :

In $\dfrac{9}{2}$ minutes it goes $\dfrac{7}{2}$ miles ;

$\therefore$,, 1 ,, ,, $\dfrac{7}{2} \times \dfrac{2}{9}$,,

$$\therefore \text{ In } \frac{117}{2} \text{ minutes it goes } \frac{7}{2} \times \frac{2}{9} \times \frac{\overset{13}{\cancel{117}}}{2} = \frac{91}{2} = 45\tfrac{1}{2} \text{ miles,}$$

which is absurdly indirect. The whole process suggests the mentality of those who, when urged for hygienic reasons to "kill that fly", proceed to breed innumerable bluebottles so as to have plenty to kill. But I would have you work for something better than a world full of dead bluebottles.

But there is one even worse vice at which I must have a tilt before I leave this question.

$$\text{In } \frac{9}{2} \text{ minutes it goes } \frac{7}{2} \text{ miles ;}$$

$$\therefore \text{ „ } 1 \text{ „ „ } \frac{7}{2} \times \frac{2}{9} \times \frac{117}{2} \text{ ;}$$

$$\therefore \text{ „ } \frac{117}{2} \text{ „ „ (and now being too idle to copy } \frac{7}{2} \times \frac{2}{9}$$

again, you merely multiply what is written above).

The second line is now as direct a lie as it is possible for a child to tell ; and anyone who can burden his soul with it is better fitted to be an election-agent than a teacher.

The Unitary Method.

Of all forms of slavery, perhaps slavery to a name is the most contemptible, but I find it very prevalent where "The Unitary Method" is concerned. The point of the method is to use a convenient stepping-stone when required. *One* is often a convenient stepping-stone : therefore call it "The Unitary Method" ; and when you have given it that name, you *must* use one as your stepping-stone, "because it is the Unitary Method".

By spending £160 he gained £72. What did he gain per cent.?
$\therefore$ „ „ £20 „ „ £9.

"No", says the boy ; "you have to say 'By spending 1'." "Why should I, if I find 20 a more convenient stepping-stone between 160 and 100?" "It's the Unitary Method ; and that means you use 1." In fact I even meet boys who object to the step from 50 to 100 on the ground that I ought to have used 1 as a stepping-stone and "put 100 over 50".

π.

I have expressed a preference for $3\frac{1}{7}$ rather than $\frac{22}{7}$ as the right name for a commonly-used approximation to π, chiefly on the ground that it carries its meaning and an idea of its size more clearly stamped on it, and that multiplication by 3 is easier than multiplication by 22. As a small side-issue arising out of this, I would have you notice how easily multiplication by $3\frac{1}{7}$ can be converted, if required, into multiplication by Adrian's much finer approximation $3\frac{16}{113}$, or $3 + \frac{1}{7}(1 - \frac{1}{113})$.

Suppose that we want $7{\cdot}29\pi$.

$$\begin{array}{r} 7{\cdot}29 \\ 3\tfrac{1}{7} \\ \hline 21{\cdot}87 \\ +\ 1{\cdot}04 \\ \hline 22{\cdot}9 \end{array}$$ ... correct to 3 figures.

Now suppose that a better approximation to $7{\cdot}29\pi$ is required.

$$\begin{array}{r} 7{\cdot}29 \\ 3\tfrac{1}{7}- \\ \hline 21{\cdot}87 \\ +\ 1{\cdot}041429- \\ -\ \ {\cdot}009216+ \\ \hline 22{\cdot}90221 \end{array}$$ ... correct to 7 figures.

(Of course if we had started to use the finer approximation from the beginning without the rougher one first, it would have been better to subtract a 113th part from the 7·29 before the division by 7, so cutting out the need for the extra figure. If you put down the working as here, but without the 8th figure, there is a chance of nearly 1 in 4 of the 7th figure's being wrong through accumulation of errors.)

The diameter of the Earth multiplied by $3\frac{1}{7}$ differs from the circumference by ten miles. Multiplied by Adrian's approximation $3\frac{16}{113}$, it differs from the circumference by eleven feet. This king of approximations deserves to be more widely used than it is : I believe that people are afraid of it because they call it $\frac{355}{113}$ (the Improper Fractions Superstition), and never learn this way of making multiplication easy. Call it three and a tinkered seventh, the tinkering consisting of decreasing it by $\frac{1}{113}$ of itself, and you will make it much more readily available for general use.

Every boy knows that Hannibal lived a long time ago, though it is not so generally known that Archimedes fought against the Romans in the same war as Hannibal ; and it is absurd that the world should be full of boys taught that " π means 22 over 7 " when Archimedes proved 22 centuries ago that π is less than $3\frac{1}{7}$. Given the diameters of the Earth, if you calculate its area taking π as $3\frac{1}{7}$, you get an answer differing from the true area by just about the area of England and Scotland, which I hope you will not class as negligible countries.

The Extra Figure Superstition.

It is well known that if you want a sum, difference or product to be certainly correct to a required number of places of decimals, you must begin by having more places of decimals before the addition, subtraction or multiplication. " The Extra Figure Superstition " is that the same principle applies to division ; and I have met examiners so convinced of this that they take off marks for a division in which

an extra figure has not been used, even when the working and answer are completely right.

Now the fact is that in division the error is not cumulative ; and in order to divide an approximate quantity by an integer, you never need to have it to more places of decimals than your final answer, nor yet to continue the division to an extra figure. I will not prove all cases of these, but only give instances and prove one or two cases. Answers are to the nearest integer : by merely shifting the decimal point, all these are applicable to any number of places of decimals.

(1) $857 \div 35$.

$$5\overline{)857}$$
$$7\overline{)171}$$
$$\underline{24}$$

(2) $811 \div 108$.

$$9\overline{)811}$$
$$12\overline{)\ 90}\ +$$
$$\underline{8}$$

(3) $\sqrt{44091} \div 84$.

$\sqrt{44091} = 210 -$

(Minus how much doesn't matter : it is enough to know which side of 210 the exact square root is.)

$$7\overline{)210} -$$
$$12\overline{)\ 30} -$$
$$\underline{2}$$

(4) $173 \times \frac{16}{113}$ [treated as $\frac{1}{7}\left(1 - \frac{1}{113}\right)$].

(In this case the 173 must be exact to make the method proof against cumulative error.)

$$173$$
$$-2$$
$$7\overline{)171}$$
$$\underline{24}$$

Proof of (1). a and b are odd integers : x is not necessarily an integer.

y is the nearest integer to x/a.

z is the nearest integer to y/b.

Required to prove that z is the nearest integer to x/ab.

$$\frac{x}{a} \sim y < \tfrac{1}{2},$$

$$bz \sim y \leqslant \frac{b-1}{2}.$$

Thus $$\frac{x}{a} \sim bz < \frac{b}{2},$$

and $$\frac{x}{ab} \sim z < \tfrac{1}{2}.$$

Hence z is the nearest integer to x/ab.

Proof of (4). (In this case x is an integer.)

y is the nearest integer to $x/113$.

z is the nearest integer to $(x-y)/7$.

Required to prove that z is the nearest integer to $\dfrac{16x}{113}$.

$$\frac{x}{113} \sim y \leqslant \tfrac{56}{113};$$

$$\therefore \left(x - \frac{x}{113}\right) \sim (x-y) \leqslant \tfrac{56}{113};$$

$$\therefore \frac{112x}{113} \sim (x-y) \leqslant \tfrac{56}{113};$$

$$\therefore \frac{16x}{113} \sim \frac{x-y}{7} \leqslant \tfrac{56}{791}.$$

But $$z \sim \frac{x-y}{7} \leqslant \tfrac{3}{7} = \tfrac{339}{791};$$

$$\therefore \frac{16x}{113} \sim z \leqslant \tfrac{395}{791} < \tfrac{1}{2};$$

$$\therefore z \text{ is the nearest integer to } \frac{16x}{113}.$$

The generalised form of this presents no difficulties, but is tedious to write and to read. In the case of an even divisor we sometimes need guidance, as in (2) and (3), whether to take the big or the little half.

What makes an Approximation "Correct"?

In all questions of an approximation correct to a certain degree of accuracy, the prominent idea should be "nearer to this than to that". "$\sqrt{10} = 3{\cdot}16\ldots$" means that $\sqrt{10}$ is nearer to 3·16 than to 3·15 or 3·17. $\dfrac{1{\cdot}7321}{3} = {\cdot}5774\ldots$, the last 4 being got by the consideration that we are dividing 11 by 3, and 11 is nearer to 12 than to 9—a procedure which I consider hugely preferable to all this slushabout with an extra figure, with rules about 5 being more than five and therefore ten. (It is well known that $\log_{10}\pi = 0{\cdot}4971499\ldots$, which to 5 places is 0·49715; and therefore, according to somebody's rule that 15 counts as 20, it is 0·4972, which is actually printed as the value in somebody's tables which are still on the market.)

An approximation is correct to as far as it goes if the exact value is nearer to it than to any other expression of the same or less degree

of complexity, which is sometimes capable of exact definition and sometimes a matter of opinion. For instance, each of these approximations to $\sqrt{\frac{1}{2}}$ is nearer to $\sqrt{\frac{1}{2}}$ than any other expression of the same sort: 1, ·7, ·71, $\frac{12}{17}$, $\frac{70}{99}$ (which is nearer to $\sqrt{\frac{1}{2}}$ than any other fraction made of smaller numbers).

Approximations of the type $\frac{70}{99}$ were more used by the ancients than by us, who have nearly killed them off with our severely tabulated decimal notation: they survive like fine but rarely worn hand-made fabrics into a pitiless machine-age; and for mental calculations especially they are much easier to use than decimals of the same degree of accuracy. For instance, find the side of a square whose diagonal is 61 metres. 61×70 is $4270 = 4300 - 30$. Divide by 99: it goes 43 times: the remainder is the difference between 43 and 30, that is 13. Divide 13 by 99 and get ·13. 43·13 metres is the length, correct to four figures.

It is interesting to make a collection of expressions like $99 - 70\sqrt{2}$, in which the squares of the two terms differ by 1, from which fractional approximations to square roots of integers may be obtained to any required degree of accuracy: I will say no more about that except that $\sqrt{46}$ is a tough nut to crack.

Remainders.

How many shares costing £$6\frac{3}{8}$ each can be bought for £300? How much money is over?

The way in which most boys do this seems to me to offend against the principles both of simplicity and truthfulness.

Number of shares bought

$$= \frac{300}{6\frac{3}{8}} \text{ (which is untrue, because this is not an integer)}$$

$$= \frac{300}{1} \times \frac{8}{51} \text{ (the Improper Fractions Superstition and the cancelling disease)}$$

$$= \frac{800}{17}$$

$$= 47\tfrac{1}{17}$$

$$17 \overline{)800} \quad 47 \quad 68 \quad 120 \quad 119 \quad 1$$

Next we take the remainder, which is posing as a fraction of a share, and convert it back again into the money which it really was all the time.

$$\tfrac{1}{17} \text{ share} = £\frac{3}{\cancel{51}} \times \frac{1}{\cancel{17}} \text{ (more cancelling)} = £\tfrac{3}{8} = 7\text{s. } 6\text{d. remainder.}$$

As a start towards doing this better, the first rule I would give is, "Call them pigs". Children have seen pigs and probably smelt them; but they are not familiar with the look or the smell of shares.

Now we will be a farmer starting out to buy pigs which cost £6⅜ each—a simple common-sense man of little education, but knowing that eight half-crowns make a pound. He has £300 in his wheelbarrow; and because the price of pigs is in eighths of a pound, he has changed it all into 2400 half-crowns.

Arrived at the market, he chooses a pig and shovels out 51 half-crowns to pay for it, chooses another and does it again, and goes on till—till what? Till he has to stop because he hasn't 51 half-crowns left in the barrow. Then what is left in it? A fraction of a pig? Bunkum: it is a few remaining half-crowns.

Now put down the working like this:

```
h.c.     47 pigs (or shares)
51 ) 2400 h.c.
     204
     ---
     360
     357
     ---
       3 over.  (3 what over?  3 of these 2400
                 things.  What are they?  Half-crowns.)
                                  7s. 6d. over.
```

Many boys will put down all this except the headings "half-crowns" or "£⅛'s"; and for want of that they don't understand the remainder 3 when they get it, but embark on another calculation so as to find it over again. Stating your unit is the key to it, and behind that lies the principle, "Think about a real farmer buying real pigs with real coins in a real wheelbarrow."

And closely allied with this principle is another: "Be a bit more generous in the use of plain English words, and avoid those with double meanings." "On" is a dangerous word with a multiple meaning. "On 70 he gains £3" may mean that he spent £70, sold for £70, or bought 70 tons or ounces or tea-leaves. "By spending" can have only one meaning, and that is the clue to all questions in which you are asked for the gain per cent.

Percentage.

We all know that Percentage is one of the most fruitful sources of confusion to the young. The veterans among you may remember "The Pinnacle Soap-dish". Its floor was made of rubber spikes like the back of a batting-glove, which supported the soap above the slush-level and saved it from degenerating into that peculiarly unpleasant pale suety texture which is the fate of soap that welters in its own slime. "The Pinnacle Soap-dish", according to its advertisements, "saved 50 per cent. of your soap-bill". But I have never met a parent who gave a really convincing answer when the young, as one man, lifted up their voice and demanded, "Why not buy two, and save the whole of the soap-bill?"

For percentage, as for so many other arithmetical diseases, my panacea is, "Be generous in the use of simple English words". I think it is a great mistake to teach boys Sherlock Holmes questions,

in which you are given what happened last in time and asked what happened first, before they are familiar with straightforward questions in which they are told what happened first. When this mistake is made, boys import the inverse methods of the backwards problem into straightforward questions, which is as logical as buying a stomach-pump to eat your breakfast with.

Here is an instance : " 22 lb. of tea at 2s. per lb. are mixed with 27 lb. at 3s. per lb. The mixture is sold at 2s. 9d. per lb. Find the gain per cent.".

22 lb. of first	tea cost	44s.
27 lb. of second	,, cost	81s.
49 lb. of mixed	tea cost	125s.

After that most boys will proceed to find in 49ths of a penny what one pound or one ounce or one tea-leaf cost ; but the few who avoid this trap and discover (probably by improper fractions or reduction to pence) that the selling price is $134\frac{3}{4}$ shillings will go on to write something like this :

" 125 is 100 (or worse still, " 125 = 100 ")

(a peculiarly unsatisfying abstraction, with neither simplicity nor truthfulness to commend it) ;

$\therefore$ 1 is $\frac{100}{125}$;

$\therefore$ $134\frac{3}{4}$ is—a large mass of slaughtered figures, resembling a portrait of a hedgehog, and leading ultimately to 107·8 ;

$\therefore$ the gain per cent. is 7·8."

My two chief objections to this are that it doesn't mean anything obvious and that it uses unnecessarily big numbers. What I want is *simple English words.*

By spending	125s.	he gained	$9\frac{3}{4}$s. ;
,,	,,	500	,,	39 ;
,,	,,	100	,,	7·8 ;

and this is the meaning of " gaining 7·8 per cent.".

(At tea-time Mr. Boon will explain to you that the percentage ought really to be taken on the selling price ; but I am following here the general practice of Arithmetic books and the principle that percentage is taken on the thing that happens first in time, as in interest and population questions. I am also assuming that time progresses in the same direction for all of us.)

Young Methods.

And now if I inflict on you another scrap of autobiography, it is not that I expect you to be interested in my life for its own sake, but because I believe you will find it in some important respects the same as your own. When I had been teaching for some years, I began to notice that I often did questions better than the boys I taught. This may have been your experience too. But whenever

you find this happening, I want you to look into the reason. You are older, probably cleverer, and have had a university education : all these things would naturally enable you to use more advanced methods than your pupils. But is that how you excel them? In my own experience exactly the opposite thing has happened : in Arithmetic at any rate, when I do a question better than a boy, I nearly always find that I have done it by a more elementary method than the boy, by an appeal to the concrete and visible, by thinking what would interest a real grocer in a real apron picking real maggots out of a real cheese, and living on real shillings got by selling real coffee. In fact a strengthened and steadied version of the eight-year-old mind is more effective than the fourteen-year-old mind bunged up with the superstitions that somebody has choked it with at school. My own progress in Arithmetic has consisted largely in forgetting dodges for making easy things difficult, and reversion to the methods of my good Scottish governess who, before I ever went to school, taught me the Battle of Bannockburrn for History and common sense for Arithmetic. Even if your own early education was conducted with less emphasis on the Battle of Bannockburrn (thirrteen-fourrteen), I would have you look at every question in Arithmetic which you do better than your pupils, and notice how often you have done it by a younger method rather than an older one. Possibly something of the same sort was in the mind of the Psalmist who wrote, " Out of the mouths of very babes and sucklings hast thou ordained strength". Another verse of his which I commend to your attention is, " Confounded be all they that worship carved images ". And that is just what they are, thoroughly confounded ; and to straighten out their confusion you must turn them away from their elaborate idols.

ADDITION AND SUBTRACTION.

$$\begin{array}{r} 4\,8\,8\,3\,1\,2 \\ +3\,0\,3\,8\,6\,8 \\ \hline 7\,9\,2\,1\,8\,0 \\ -1\,6\,0\,5\,9\,6 \\ \hline 6\,3\,1\,5\,8\,4 \end{array}$$

In this addition there has been one to carry three times in the six columns ; and in the subtraction that follows one has been borrowed in three of the six columns. But if the two operations are combined together into one, four of the six columns are complete in themselves, in one we have borrowed, and in one we have carried.

$$\begin{array}{r} 4\,8\,8\,3\,1\,2 \\ +3\,0\,3\,8\,6\,8 \\ -1\,6\,0\,5\,9\,6 \\ \hline 6\,3\,1\,5\,8\,4 \end{array}$$

These are the actual expectations, and it can be proved that, whatever the scale of notation, the probability of a borrow in any

column is $\frac{1}{6}$, of a carry $\frac{1}{6}$, and a $\frac{2}{3}$ chance of a column complete in itself. This makes mixed addition and subtraction of this sort, with two positive lines and one negative, easier than the pair of operations which it replaces, and in fact much easier even than the addition of three positive numbers, in which a column complete in itself occurs only once in six times.

The same method is possible when two of these three numbers are negative ; but in that case I have not found it as easy as the method of making two bites at it, chiefly because only one column in six is complete in itself, and the borrowing becomes a strain on the memory.

The Use of Tables.

When boys reach the stage of using tables, there is much to be said for the practice of trying to get right results from them rather than wrong ones. And for this the first requirement is to know the history of Jimmy and Sally.

Once upon a time there was a little boy and his name was Jimmy ; and he lived in a house on a long, long road with milestones all along it, placed as accurately as a professional surveyor could place them. Buses ran along the road, but they wouldn't stop except at the milestones.

One day Jimmy's mother said to him : " Jimmy," she said, " go and see your grandmother : she lives 100·9 miles away along the road ". So she did up two ham sandwiches and a piece of currant cake in a brown paper parcel for him, and put him into a bus, and paid his fare and asked the conductor to take great care of him ; and the bus started.

When the bus got to milestone 100, Jimmy got out and began to try and pace another $\frac{9}{10}$ of a mile. But he wasn't very good at pacing, and his little legs were rather podgy : so by the time he had paced $\frac{9}{10}$ of a mile he was a good deal out of his reckoning. He went in through the wrong gap in the hedge ; and instead of reaching his grandmother's house he got to a house that was full of big, black, bearded, Bulgarian brigands. When they saw little Jimmy coming up the garden path, they laughed a hoarse laugh, " Ha-ha " ; and one of them took a long knife out of his boot, and they took his little collar off him and cut him up into vermicelli and stewed him with onions for their supper. And that was the end of Jimmy who tried to pace $\frac{9}{10}$ of a mile.

About a week later, Jimmy's mother said to his little sister Sally : " Sally," she said, " go and visit your grandmother : she lives in a house 100·9 miles away along the road ". So she did up one ham sandwich and a piece of the best currant cake in brown paper, and put Sally in the bus, and paid only half-fare for her because she was under 12, and was so flustered that she nearly kissed the conductor instead of Sally ; and the bus started.

Now Sally was a thoughtful child ; and when she got to milestone 100 she didn't get out, " because ", she said, " my granny lives

nearer to milestone 101 ". So she got out at 101 and began to pace back one-tenth of a mile towards home. Now she wasn't really any better at pacing than Jimmy ; but because she had only one-tenth of a mile to pace, she didn't go anything like as wrong as Jimmy : so she got to her grandmother's house, and her grandmother was awfully pleased to see her ; and they had prawns and lemonade for supper, and a lovely tall white creamy pudding with almonds stuck into it all over to make it look like a porcupine.

				Mean Differences.			
	0	1	.....	1	.....	9	
10	0000	0043	.....	4	.....	37	

Now if you will look out the Jimmylog of 100·9 in the four-figure tables which you have so thoughtfully brought with you, you will see that it comes to 2·0037 (or 8 if you have a split-line table), but the Sallylog, obtained by getting out at the nearest milestone, 101, and then pacing back $\frac{1}{10}$, is 2·0039. As these are different, they can't all be right ; and as the seven-figure table gives 2·0038912, there is no doubt which of them is the truth.

" And the moral of that ", as the Duchess would have said, " is ' Always get out at the nearest stop '." In fact, completely abolish all mean differences for more than half the interval tabulated, and use the small ones only, getting out at the nearest carefully-surveyed milestone and doing the smallest possible quantity of the most inaccurate work, this interpolation which I have compared to pacing.

Here is a line out of a table of logarithms of sines of angles, arranged by six-minute intervals, as is usual, and giving mean differences up to 5 minutes.

A LINE OF THE LOG SIN TABLE, WITH 6′ INTERVALS.

											Mean diff.				
	00′	06′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
8°	$\bar{1}$.1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42

Here, for comparison, is the corresponding line of a table in which the interval is five minutes, and mean differences are given for 1′ and 2′ only. It is written on the same scale as the other, and you will notice that it is exactly the same length.

A LINE OF THE LOG SIN TABLE, WITH 5′ INTERVALS.

													M.D.	
	00′	05′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	1′	2′
8°	$\bar{1}$.1436	1480	1525	1568	1612	1655	1697	1739	1781	1822	1863	1903	8	17

I have looked out the log sin of every minute from 8° 01′ to 8° 59′ in three ways : (*A*) by using the 6′ table and always adding

the mean differences; (*B*) by using the 6′ table and small mean differences only, subtracting when necessary; (*C*) by using the 5′ table and small mean differences for 1′ and 2′ only. The next graph shows the comparative accuracy of the three methods by giving you the frequency-distributions of the errors in the 4th figure. (No distinction is made here between positive and negative errors: they are grouped by absolute magnitude only.) The sizes of the errors are measured from left to right; and up the page is the number of times that errors of each size occur.

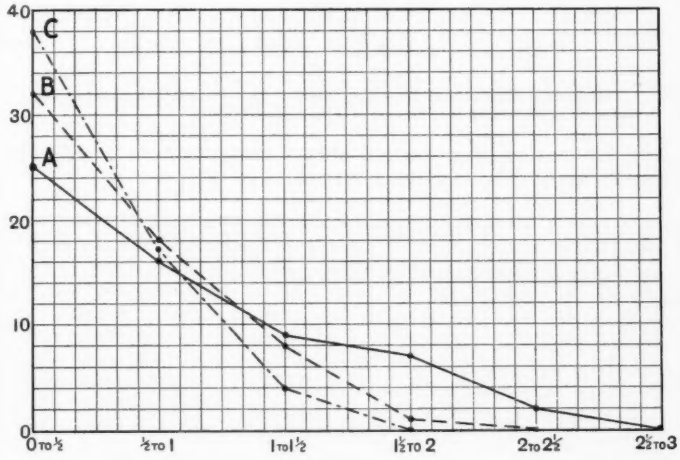

FIG. 1.

Graphs of frequency of errors in 4th figure of log sin $(8° + n')$, obtained
(*A*) by 6′ intervals and addition of all mean differences (——)
(*B*) by 6′ intervals and use of small mean differences only (— — —)
(*C*) by 5′ intervals and use of small mean differences only (— - — -)

The Jimmy method *A*, in which big mean differences are used, is shown to be very inaccurate by the relative frequency of the big errors, some even more than two. *B*, in which Sally's principle is followed, and the big mean differences for 4′ and 5′ are never used, is much more accurate, as is shown by the rarity of the bigger errors. In four cases out of six, methods *A* and *B* are identical, and therefore their disparity is here shown greatly diluted, and should be multiplied by three to show what really happens when the distinctively Sally principle is in action.

The graph *C* is got from the table with 5′ intervals, using small mean differences only. System *C* gives no errors more than $1\frac{1}{2}$: it is incomparably better than *A*, and is got from a table with lines of the same length. Table-makers, please note. Notice also the great advantage of an odd number of minutes in the interval:

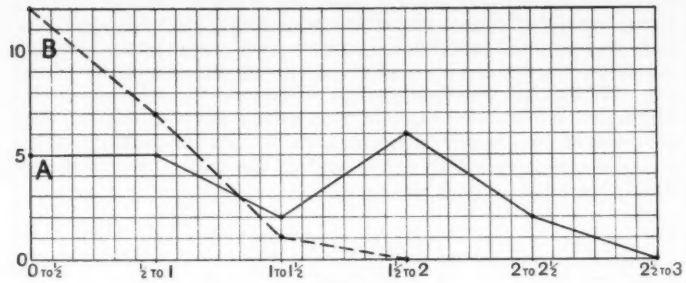

FIG. 2.

Frequency-distribution for those cases only in which the number of minutes is $6n+(4 \text{ or } 5)$. Interpret A and B as before.

with an even number you are always being confronted with two equally likely values for things like log sin 8° 27′.

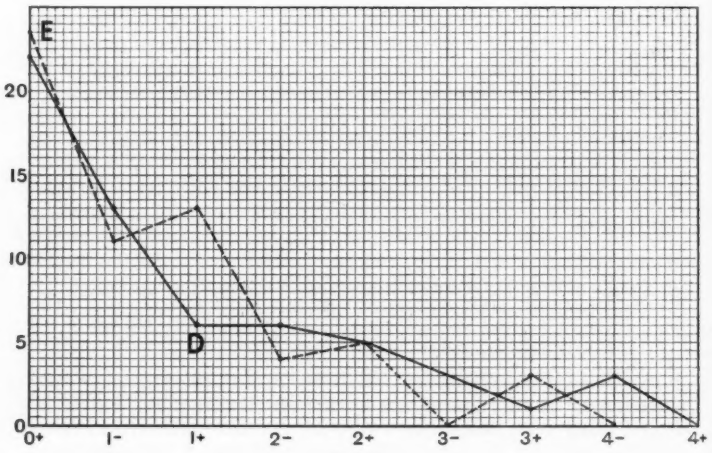

FIG. 3.

Frequency-distributions of errors in the 4th figure of log sin $(8°+n')$, obtained
(D) by 10′ interval tables, adding mean differences up to 9′ (——).
(E) by 15′ interval tables, using mean differences up to 7′ (— — —).

If you look at these two frequency-distributions of errors, it is not very obvious which is the better. I think I prefer E to D on the whole, as giving higher frequency to the smaller errors and lower to the bigger ones than D. But D is got from the Cambridge University Table, with 10′ intervals and using mean differences up to 9′,

and E from a regular cad table, with 15′ intervals, but using no mean differences above 7′. The odd number of minutes and the expurgation of the foulest mean differences are enough to compensate the greater length of the interval.

By choosing a line of the table in which the mean differences behave even worse, I could have made the disparity between the good method and the bad much greater; but here I have taken a line in which the mean differences are good enough to use for most purposes. For very small angles they are not, and that is why the four-figure tables which I use most give the log sin of every single minute, and not only every 6th minute, up to 8°.

Where shall Wisdom be found?

It only remains to tell you where to find the boys who write clean, sensible, checkable working. Nearly twenty years ago I examined about my own weight of Arithmetic papers for the Cambridge Local. They came from every latitude and longitude, and were done by children of every colour of the rainbow and the coal-mine. Good papers came from West Africa, done by Macaulay Babington Gladstone and O. O. Onabanjo, and delightful papers from Mauritius in broken English that was half French. But the gem of the collection was a batch of twenty supremely beautiful papers from Penang. Most of them were up to the Distinction standard; and every one of them was a work of art, a joy to the eye and a rest to the brain. Every question was easier to check by reading it than by doing it again from the beginning. The boys who did them had names that would sound queer to you, Chŋg Kah Sim, Ng Ching Choon, and (particularly endeared to me by his second name) Chew Boon Huck. But to my mind the important point is this: there is already in the world at least one place where Arithmetic is a thing of beauty and a joy to its doer and its corrector; and when once a man has seen the kind of work they do in Penang, it is impossible to persuade him again that Arithmetic need be the depressing slush that we make of it in this country. Therefore let us either teach our children to write clean sense in simple English words, or let us charter a large ship and invite the entire population of Penang to get aboard of her, come to this dark island of superstitions, slaughter and lies, and undertake the gigantic task of our education. W. H.-J.

1240. Lilliput and Brobdingnag teach many lessons. The one is feasible, the other utterly impossible mechanically; but whether human intelligence could ever fit into a Lilliputian brain is another question. Swift's arithmetic is of the soundest when he gives Gulliver 1728 Lilliputian rations, and just one Lilliputian hogshead for his glass of wine; while Gilbert White of Selborne, arguing that a flamingo's legs should be longer than a lesser bird's in proportion to the weights of the two, displays complete ignorance of mathematics. But physiology is more subtle than arithmetic; and 1728 Lilliputian dinners, arithmetically correct, would be far too much for Gulliver.—Sir D'Arcy Thompson. From a letter in *The Times*, December 10, 1938. [Per Mr. A. F. Mackenzie.]

MATHEMATICS IN SCHOOL AND UNIVERSITY.*

By G. B. Jeffery.

We have all heard the scheme recommended to Headmasters for their guidance in making the first three reports on a new pupil. The first should be devastating in order to impress the parent with the toughness of the job that the school is up against in remedying the defects in the earlier training of the pupil. The second should be glowing in order to show the progress made when the boy gets to the right school. The third should be very guarded in order to moderate any undue optimism created by the second, and to make it clear that any failure that may lie in the future will be due solely to lack of aptitude or application on the part of the pupil. I fear that some of my colleagues in the universities are prone to adopt a similar line of defence. Certainly one not infrequently hears in college common rooms of the shockingly incompetent teaching given in the schools in this subject or in that. One may sometimes suspect that this is a professional particularisation of the proposition that "all good workmen have bad tools", though every beginner in logic knows that this may not be simply converted to "all who have bad tools are good workmen". Even if the teachers are as bad as some would have us believe, we professors who have had a hand in the training of so many of them had better keep rather quiet about it.

My own opinion, based on observation of the quality of the students who come up to the Universities and on the opportunities I have had of observing the school examinations at close quarters, is that the teaching of mathematics has made very great progress since I was at school, and is now so good that a university professor should criticise it only with considerable hesitation and with some modesty.

It is a great merit of our Association that it brings together those who are concerned with the teaching of mathematics to pupils of different ages and different stages of development in the subject. It helps us to realise, at least within the limits of our own subject, the ideal which many of us have set before us—a unified teaching profession embracing colleges and universities and schools of all grades and kinds—a profession in which the importance of the work of the individual will no longer be measured by the average age of his pupils. Are we not constantly discovering that the special difficulties and problems that arise in the course of our work arise in a modified form in the work of those who are teaching pupils of a different age?

The great merit of a Presidential Address from the point of view of the President is that he is permitted to rove over the whole field of our subject in a way that would not be tolerated in a communication from an ordinary member. I propose to avail myself fully of this privilege. I have no great theme to offer for your consideration,

* Presidential Address to the London Branch of the Mathematical Association, 15th October, 1938.

but only a few thoughts of no great profundity on the value of our subject to our pupils.

Can we justify the place that mathematics has hitherto occupied in the school curriculum? In the past every boy and girl has spent a quite considerable part of the school time on mathematics of some sort, from the very earliest stages at least up to the School Certificate stage. We have been a sheltered industry teaching a compulsory subject, and we have not been called upon to justify our existence as teachers of other subjects have had to do. It is long since this phase was passed in the universities; now perhaps it is passing in the schools. Next year it will be possible for the first time to take a School Certificate Examination which does not include mathematics, not even a paper on arithmetic. Are we as teachers of mathematics to regard this as loss or gain? Certainly it is a challenge to us to consider anew why we teach mathematics, and in what way we think that the study of our subject is profitable to our pupils.

The old arguments no longer carry conviction. We should like to think with Robert Recorde that Arithmetic is the "Whetstone of Witte". It would be very flattering to those of us who have spent many years in the study of mathematics if we could believe that our minds have been trained and our intellects have acquired a penetrating quality that is denied to those who study lesser subjects. Alas! that this comforting hypothesis should fail so utterly under the test of experience. Are we mathematicians conspicuous among our fellows for the clarity of our thinking on, say, politics? Do the statesmen turn to us to resolve the tangled problems of Europe? They do not, for there is a limit even to the folly of politicians.

Yet I understand that the psychologists permit us to save something from the wreck. There is some gain in learning to think clearly about something. There is at least a faint hope that we may be a little more able and a little more anxious to think clearly about the next thing that comes along.

If that hope is justified, then it seems to me that mathematics is a particularly favourable field for its exploitation. Mathematics is usually regarded as a difficult subject, and in some ways it is, but its difficulty arises from the complexity of the patterns it weaves on a simple loom. One has to seek far in mathematics to find an argument that is intrinsically difficult. The arguments are often long sustained. It sometimes requires great ingenuity to construct the argument which bridges the gap between the data and the desired conclusion. But once it is marshalled the argument consists of steps, each of extreme simplicity. The things with which mathematics deals, numbers, quantities, straight lines, curves, are simple to the pure in heart. I know that they are by no means simple to the mathematical philosopher, but his is a different problem, and the difficulties he discovers belong to an order of which, fortunately for him, the schoolboy will remain in ignorance for a good few years at least. To the schoolboy the things of mathematics are simple ideas illustrated on every hand in his everyday life. Moreover, the

assertions of mathematics are susceptible of clear and concise statement.

Hence it seems to me that mathematics gives us the opportunity of thinking clearly and in a simple way about some of the simplest notions that the human mind possesses. It is the superlative introduction to the art of clear thinking. This seems to me to be the most important value of geometry at the school stage, and I make one practical inference from this, namely, that work in geometry should be unhurried and that the time element ought not to bulk largely in a geometry examination paper. Different elements contribute to the value of arithmetic and algebra. The boy who can make a calculation correctly in five minutes is better than the boy who takes fifteen minutes for the job, and the examination paper may rightly test both accuracy and speed. I do not think that the same is true of geometry to anything like the same extent. A rider is an exercise in clear thought, and most of us find it difficult to think clearly if we are hurried and pressed for time. I am much more interested to know whether the boy can solve the rider given all reasonable leisure than I am in knowing how many he can solve in a given time. It is very easy for the examiner to use the time factor to produce the desired mark distribution, and thus to produce a paper which tests the wrong thing. I think that most School Certificate geometry papers could be shortened with advantage and less required in the time. If the mark distribution went wrong no great harm would be done to anyone.

Clarity of thought is closely related to neatness and clarity of expression on paper. My own standards in this respect have deteriorated sadly from what they once were, yet whenever I get tied up with a mathematical problem I take a clean sheet and start again in my best handwriting and neatest figures. Though the best handwriting is now rather bad and the figures are not as neat as they were, the old method still works surprisingly well. I am glad that this is fully recognised in the schools. From this point of view the work done for the School Certificate Examination is, with rare exceptions, altogether excellent. It shows a marked contrast with the work of the same standard done at the Matriculation Examination. Unfortunately it represents a habit that can be lost. The same candidates appear at other examinations a few years later, and all effort for neatness and decent presentation has disappeared. In fact it seems to be roughly true that the more advanced an examination in mathematics, the more untidy, disorderly and illegible the work will be. No doubt this is due to the fact that most examinations in mathematics test speed as much as they test anything. The man who wants to do well in them must sacrifice everything to speed, and he soon acquires the habits of the overhasty.

We may justify our subject on the ground of its utility. Mathematics is useful knowledge. But he who employs a utilitarian argument must be prepared for the next question—useful for what? Part of the answer is easy to give and is very weighty and convinc-

ing. In our industrial and mechanised age some knowledge of mathematics is useful in very many walks of life—in science, economics, engineering, accountancy, and finance. The number of ways in which mathematics enters into modern life is so great and is increasing so rapidly that this in itself is a justification for giving an important place to the subject in the school curriculum. The future career of the boy may not yet be planned, but the chance that it will lie in a direction in which some mathematical equipment will be useful to him is quite high.

We need not adopt a purely vocational view of education in order to justify our subject. If we conceive the business of education to be to make the child at home in the world in which he finds himself, then some knowledge of mathematical things is essential. When we first begin to interpret our sense perceptions in terms of objects situated in space we are recapitulating one of the most practically important discoveries of the human mind—the discovery of space as a means of co-ordinating our perceptions and presenting them to our minds in a comprehensible form. When we learn to count we are repeating for ourselves another of the great discoveries of our ancestors. If we try to clear our minds of every geometrical or arithmetical notion, and then try to think of anything at all, we soon discover how fundamental mathematics is in our common everyday thought. The mathematics we discover in this way is not, of course, the artificial mathematics of the textbook. It is a kind of " natural " mathematics, and the closer we can keep to this natural mathematics in our teaching of the beginners the more valuable our subject will be as an instrument of general education.

There are two conclusions that I draw from this view. The first is as to the extreme importance of the right teaching of very young children on these matters. We who are concerned with older pupils are sometimes impatient of the great care that is now given to the first steps in number work, and to the geometry of shapes and objects taught at the kindergarten stage. Yet it seems to me that careful work at this stage may well be a determining factor in the future development of the child, and that if this work is rushed by the expedient of supplying the child with a set of ready-made rules we are permitting a structural defect in the very foundations of our work.

My second conclusion is that simple three-dimensional geometry should be given a larger place in our school work, for it is three-dimensional space rather than two-dimensional that we discover through our perceptions. One of the best ways of learning the simple geometry I have in mind is through handicraft. The boy who can plane up a piece of wood and test it for winding and squareness has grasped in a particularly vivid way the substance of half-a-dozen significant propositions in solid geometry. The mathematical teacher will find the teacher of handicraft a useful ally in other ways. A good cabinetmaker has a fund of geometrical knowledge. Practice in the making and in the interpretation of working drawings brings a practical appreciation of much good geometry. Quite

apart from its other values a craft which, like woodwork, admits of considerable precision in shape and measurement, is a valuable means of acquiring a grasp of geometrical truth.

Another ground for our faith in the value of our subject is to be found in what I believe to be a fact, namely, that a considerable proportion of our pupils enjoy mathematics. The view of education as a preparation for life sometimes leads us to assume that life begins at the school-leaving age. It is by no means clear that something which contributes to the enjoyment of life at the age of sixteen is on that account less valuable than something which contributes to the enjoyment of life at fifty. And so the mere fact that so many of our pupils enjoy mathematics seems to me to be part of the justification for teaching it. Even so, in a way we perhaps do not fully understand, we are helping the pupil to prepare for his later life. Just as free and enjoyable activity of the body provides a particularly beneficent form of physical exercise, so free and enjoyable activity of the mind seems to contribute to our intellectual growth.

It is perhaps worth asking why boys and girls of school age should find enjoyment in mathematics. The joy of mathematics is twofold in its character. It arises partly from the artistic satisfaction we have in contemplating mathematical results. I have in mind the great theorems of pure geometry and some of the broad and fundamental theorems of analysis. Of the same kind is the satisfaction we find in the beauty of form in argument—the delicate touch and the ingenious turn that show the master mind. Now I do not think that there is very much in school mathematics that has this quality of mathematical beauty. Perhaps we come nearest to it in geometry in such simple results as this, that no matter what triangle I take its medians are concurrent. In my own mind this quality is usually associated with great generality, and a high degree of generality is usually inappropriate to the school stage.

But there is another source of satisfaction in mathematics that is more nearly akin to the crafts—the satisfaction of attempting something, and doing it, and knowing that one has done it. Mathematics presents opportunities of this sort more freely than most other subjects of the curriculum. A mathematical problem presents a short range objective ; you either solve it or you do not ; and when you have solved it, you are usually quite certain that you have solved it, though you may occasionally be mistaken. On the other hand, the answer to a question on history may have any value between perfection and complete inaccuracy or error. The value of the answer is relative to the stage of development of the pupil. A question could be set in identical terms at a school certificate examination and at a degree examination. An answer which would be regarded as approaching perfection in the one case would be quite inadequate in the other.

This characteristic of mathematics creates a special problem in the conduct of examinations. It is easy to frame a paper for a competitive examination in mathematics. All one has to do is to include

a certain number of questions which are beyond the powers of the weaker candidates and a certain number that only the really brilliant ones can tackle. The circumstances of a School Certificate Examination are quite different. Here every question is within the powers of the average well-prepared candidate. It is therefore to be expected that a considerable proportion of the candidates will obtain substantially full marks. This would not conform with the normal distribution nicely balanced about 50 per cent., which is so dear to the heart of some examining bodies. As far as mathematics is concerned this seems to be a quite arbitrary fashion imported from other subjects in which it may have some justification, and often too lightly accepted by mathematical examiners.

In fact the mark distribution in mathematics does not differ very much from that in other subjects. This result is achieved by framing the examination so as to test two elements distinct from mathematical skill and knowledge. Firstly, the examination is used to test powers of presentation, though in a restricted and somewhat artificial way. A substantially successful answer to a geometry question may receive less than full marks because the candidate has omitted a step. In such a case we are not deducting marks because of something the candidate does not know. On the contrary, there is a presumption that he does know it, otherwise he could not have solved the problem. We are deducting marks because of something which the candidate ought to have known and stated, but which he knew and did not state. Now after what I have said in another connection you will not expect me to rebel very violently against this method of adjusting the mark distribution. From the educational point of view the art of the clear presentation of mathematical results is at least as important as the acquisition of mathematical knowledge. If, however, we are going to reflect this view in the examination we should take into account the orderliness and clarity of the whole presentation, and not concentrate merely on the inclusion of all necessary details.

The second method of adjusting the mark distribution is much more powerful, and consists in operating the time factor. If the average mark runs higher than we expect, we ask the candidates to do more in the time. The method is fatally powerful and easy to apply, and in my opinion examiners have come to rely upon it to far too great an extent. A very high premium is placed on speed, and though speed has its value in mathematics, its undue cultivation is inimical to the deeper values of the subject.

Special problems are presented by the boys and girls who are either very good or very bad at our subject. Most boys and girls of normal intelligence who are prepared to work with that moderate degree of application appropriate to healthy young persons of their age find that our subject comes quite naturally to them. There are others who find it difficult, and who progress only at the cost of really hard work. There are some who, whatever may be the reason, find the subject of extreme difficulty and make very little

progress in it. In the school examinations we have so far endeavoured to meet this situation without completely abandoning the idea that mathematics should be a compulsory subject. It has been permissible to offer a qualifying paper in arithmetic instead of the full subject. This device has no doubt effected a partial solution, but it is disconcerting to find that there are still a certain number of candidates who fail hopelessly on even this modified test, but who nevertheless show distinct merit in other directions.

We shall probably all agree that a great deal of time is wasted by boys and girls on subjects at which they are no good and never will be any good, and that it would be to the advantage of everybody concerned if they were allowed to turn their attention to other subjects in which their labours might be more profitable. The problem is, however, not quite so simple as that, for there is a value in the discipline of relatively uncongenial subjects. We have to learn sooner or later that we cannot go through life devoting our whole time to enjoyable tasks. Most of us have to spend some part of our time on jobs that we do not find particularly enjoyable, and if we are wise we get them done without making too much fuss. A boy who finds a subject difficult, but who sticks to it and wins through to some measure of success, thereby gains a great deal. It is a difficult and individual problem to decide whether the time has come for a boy to drop a difficult subject and to turn his energies in other directions, or whether he will gain more by sticking to it. If the examining bodies now abolish mathematics as a compulsory subject for the School Certificate, they are not solving this problem ; they are passing it on to the schools, and I for one believe that it is only within the school that the problem can be rightly solved.

Our very good pupils present us with a problem of a different kind, the problem of specialisation. I think that a great deal of nonsense is talked on this subject. With some " specialisation " is a term of abuse, and is contrasted with " broad education ", which on examination is usually found to be heavily over-weighted on the linguistic side. If specialisation means that each of us should discover his special gifts and then focus his activity in the direction of those gifts, then one might well claim that specialisation was one of the most important functions of education. All the same I think that specialisation in mathematics at the school stage can be excessive, and in any case should be undertaken with caution. As a professional mathematician I feel that I must always be on my guard against the essential limitations of my subject—a mathematician tends to live in a black and white world of propositions, each of which is either true or false. If from time to time he ventures to peep out into the real world he finds it extraordinarily difficult to frame any proposition which is either completely true or entirely false. His subject is essentially an abstraction. He must never forget that it is an abstraction, and how much it leaves out of account. Hence it seems to me essential that a mathematician should cultivate and keep alive some other interest, secondary per-

haps to his main interest, but effective for the purpose of keeping him in touch with the real world of people and things.

Side by side with this conviction I hold another as to the extreme value of the time spent by a boy or girl in the sixth form. It is a time of great development and opportunity. So many subjects can be treated in a way that is impossible lower down the school. That these invaluable years should sometimes be spent on mathematics to the exclusion of all else is to me the unforgivable sin of education. I know the pressure of the university scholarship system, but this is not a matter on which the schools should leave the last word to the universities. I know that some schools make a genuine effort to meet the wider needs of their mathematical specialists. I know that many more cover their scheme by a feeble pretence of a weekly essay, or an odd class in German, that fades out rapidly as the scholarship examination approaches. I would far sooner have a boy with a three-subject Higher School Certificate from a school which encourages hobbies and crafts than the typical product of the scholarship class. He has a better foundation on which to take a university course in mathematics. He may know less mathematics, but he has a better furnished mind.

After the manner of those who prefer the minding of other people's business to their own, I have spent most of my time and all your patience in speaking of mathematics in the schools. May I excuse myself by recalling what I said at the beginning of this address as to the close resemblance of our problems. Looking back over what I have written, I see nothing that I have said about school mathematics and school examinations that could not be said of university mathematics and university examinations.

There is one difficulty which I believe is more acutely felt in the universities than in the schools. It is in connection with the selection of the mathematical topics to be included in any particular course. There is a tendency to think of mathematical knowledge as arranged in a sequence. In any undergraduate course you begin at the beginning, and get as far as you can in the time available. In a growing subject any such procedure is bound to lead to an ever-widening gap between undergraduate teaching and current mathematical thought. That gap is much wider than it ought to be. We should be more ready than we are to revise our schedules frequently, to scrap the things whose importance has diminished and to include things which are more significant in relation to the present state of the subject. One has only to glance at the chapter headings in the best school textbooks of to-day and compare them with those of the corresponding books of a generation ago to see that the teachers of mathematics in the schools are alive to the necessity of this kind of adjustment.

Perhaps this problem is felt most acutely in the universities in connection with mathematics as part of a three-subject degree in Arts or Science. Very often the schedules have been conceived as the first part of a full honours course, and consist of the more elementary parts of Algebra, Calculus, Geometry, Statics and

Dynamics. The development of sixth form mathematics has robbed this work of a great deal of its interest. For the undergraduate of a generation ago this course had a large content of new and important ideas. Now it would hardly be an exaggeration to say that the student meets no mathematical idea of major importance for the first time in this course. His time is largely spent in working harder problems on his sixth form work. Neither is the content of the mathematical course particularly well adapted to the modern study of the mathematical aspects of science. It ought to be possible to devise a course on mathematics in a three-subject degree which puts less emphasis on manipulation and more on ideas. It is not easy to see how to do this in practice, and as far as I know the problem is unsolved in the English universities.

May I conclude by saying something about the transition from school to university mathematics. I have already indicated that in my view the best preparation for an honours degree course in mathematics is a Higher School Certificate including other subjects. As a general rule students who have this kind of preparation do best at the university. There are, however, some exceptions, and one meets cases in which a student with a good school record in the subject fails conspicuously at the university, while sometimes the reverse holds good. No doubt most of these cases are to be explained in terms of the development of the individual, but I think that there is a difference of emphasis between school mathematics and university mathematics which almost has the effect of making them different subjects. I have already referred to the artistic and the craft elements in the appreciation of mathematics, and said that for good reasons it is to be expected that the craft element should predominate at the school stage. At the university the emphasis shifts and we are more concerned with the logical and philosophical aspects of our subject. We are entitled to hope that the craftsman is an artist and that the artist is a craftsman, and usually it turns out so, but there is a lesser kind of craftsmanship which never develops into artistry. That, I believe, is the secret of most of our disappointments among undergraduates.

I do not think that this difficulty can be resolved by any adjustment of syllabuses. I think it can best be met by advice and guidance in the schools quite outside the framework of examinations. In advising a boy to embark on a mathematics course at a university, the master should try to see beyond the mere cleverness that can carry a boy far in school mathematics. He should try to discover whether he has a real feeling for the subject and the beginnings of that appreciation of that underlying spirit which makes mathematics something greater than all its tricks.

We can face all these problems with a cheerful confidence. There never has been a closer co-operation between school and university or a more lively consideration of their common problems than we have to-day. We may find a reasonable pride in the part that our own Association is playing in this good work. G. B. JEFFERY.

THE PROOF OF THE FORMULA FOR THE VECTOR TRIPLE PRODUCT.

By S. Chapman and E. A. Milne.

1. This note relates to the proof of the formula for the vector triple product (or continued vector product)

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where the sign $\wedge$ denotes vector multiplication and a dot denotes scalar multiplication.

2. Textbooks on vectors often begin the proof by pointing out that $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is perpendicular to its factors $\mathbf{a}$ and $\mathbf{b}\wedge\mathbf{c}$, and since the latter is normal to the plane of $\mathbf{b}$ and $\mathbf{c}$, therefore $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ lies in this plane, and must be expressible as the sum of components parallel to $\mathbf{b}$ and $\mathbf{c}$, *i.e.*

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=p\mathbf{b}+q\mathbf{c}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where p and q are scalars. Moreover, since $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is perpendicular to $\mathbf{a}$, its scalar product with $\mathbf{a}$ is zero, so that by (2)

$$p(\mathbf{a}\,.\,\mathbf{b})+q(\mathbf{a}\,.\,\mathbf{c})=0.$$

Hence

$$p/(\mathbf{a}\,.\,\mathbf{c})=-q/(\mathbf{a}\,.\,\mathbf{b})=\lambda,$$

say, and therefore

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=\lambda_{\mathbf{a};\,\mathbf{b},\,\mathbf{c}}[(\mathbf{a}\,.\,\mathbf{c})\mathbf{b}-(\mathbf{a}\,.\,\mathbf{b})\mathbf{c}], \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where the suffixes are added to λ because at this stage in the proof λ cannot be assumed independent of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The order of these suffixes, also, must for the present be supposed significant.

3. Clearly λ cannot depend on the *magnitudes* of $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$, since if any of these is multiplied by a scalar constant k, this appears explicitly on both sides of (3) and λ is unaffected. Further argument is required, however, to prove that λ is completely independent of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and that in fact $\lambda=1$.

4. It is chiefly in this part of the proof that textbooks differ. Sometimes one or more of the vectors is resolved along axes specially related to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; another method is to express first $\mathbf{b}\wedge\mathbf{c}$, and then $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ in terms of their components along three general orthogonal axes, when it at once becomes clear that $\lambda=1$. Perhaps the latter method is the most simple and the most suitable for students making their first acquaintance with vector algebra. But the use of either general or special axes, merely "for the duration of the proof", is contrary to the spirit of vector methods, and one's aesthetic sense impels a search for a method depending only on general rules of the vector calculus, proved prior (in the usual order of development of the subject) to consideration of the triple vector product.

5. Whilst reading the excellent chapter (II) on vectors in Prof. L. M. Milne-Thomson's *Theoretical Hydrodynamics*, one of us (S. C.)

was led to reconsider the matter, and as a result arrived at the proof contained in § 8 below. The proof contained in § 6 was constructed by E. A. M. with a similar object several years ago. As these proofs are to the best of our knowledge new, we hope that they may interest others and perhaps lead to the publication of different and better proofs.*

6. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be arbitrary vectors ; consider three equations of the type (3) in which $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_1+\mathbf{a}_2$ respectively are substituted for $\mathbf{a}$. Add the two first equations, and subtract the third ; in the resulting equality, the left-hand side is zero ; the right-hand side may be abbreviated by putting

$$\mathbf{X} \equiv \lambda_{\mathbf{a}_1;\ \mathbf{b},\mathbf{c}}\,\mathbf{a}_1 + \lambda_{\mathbf{a}_2;\ \mathbf{b},\mathbf{c}}\,\mathbf{a}_2 - \lambda_{\mathbf{a}_1+\mathbf{a}_2;\ \mathbf{b},\mathbf{c}}(\mathbf{a}_1+\mathbf{a}_2).$$

We find then $$0 = (\mathbf{X}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{X}\,.\,\mathbf{b})\mathbf{c},$$

or $$(\mathbf{X}\,.\,\mathbf{c})\,\mathbf{b} = (\mathbf{X}\,.\,\mathbf{b})\mathbf{c}.$$

Since in general $\mathbf{b}$ and $\mathbf{c}$ are in different directions, and not zero, it follows that $\mathbf{X}\,.\,\mathbf{b}=0$ and $\mathbf{X}\,.\,\mathbf{c}=0$, so that either $\mathbf{X}$ is zero, or $\mathbf{X}$ is perpendicular both to $\mathbf{b}$ and to $\mathbf{c}$. Since $\mathbf{X}$ is a linear combination of the arbitrary vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, it will not in general be normal to $\mathbf{b}$ and $\mathbf{c}$; hence $\mathbf{X}$ is zero. Hence

$$(\lambda_{\mathbf{a}_1;\ \mathbf{b},\mathbf{c}} - \lambda_{\mathbf{a}_1+\mathbf{a}_2;\ \mathbf{b},\mathbf{c}})\,\mathbf{a}_1 = -(\lambda_{\mathbf{a}_2;\ \mathbf{b},\mathbf{c}} - \lambda_{\mathbf{a}_1+\mathbf{a}_2;\ \mathbf{b},\mathbf{c}})\,\mathbf{a}_2. \quad \ldots\ldots\ldots(4)$$

But $\mathbf{a}_1$ and $\mathbf{a}_2$ are in general in different directions, so that the coefficients of $\mathbf{a}_1$ and $\mathbf{a}_2$ in (4) must be zero. Hence

$$\lambda_{\mathbf{a}_1;\ \mathbf{b},\mathbf{c}} = \lambda_{\mathbf{a}_1+\mathbf{a}_2;\ \mathbf{b},\mathbf{c}} = \lambda_{\mathbf{a}_2;\ \mathbf{b},\mathbf{c}}.$$

Hence λ is independent of its first suffix $\mathbf{a}$. We can therefore replace the symbol $\lambda_{\mathbf{a};\ \mathbf{b},\mathbf{c}}$ by $\lambda_{\mathbf{b},\mathbf{c}}$.

Now proceed as before with three equations of the type (3) in which $\mathbf{b}$ is replaced respectively by two arbitrary vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and by their sum $\mathbf{b}_1+\mathbf{b}_2$. Writing

$$\mathbf{Y} \equiv \lambda_{\mathbf{b}_1,\mathbf{c}}\,\mathbf{b}_1 + \lambda_{\mathbf{b}_2,\mathbf{c}}\,\mathbf{b}_2 - \lambda_{\mathbf{b}_1+\mathbf{b}_2,\mathbf{c}}(\mathbf{b}_1+\mathbf{b}_2), \quad \ldots\ldots\ldots\ldots\ldots(5)$$

we can express the resulting equality in the form

$$(\mathbf{a}\,.\,\mathbf{c})\,\mathbf{Y} = (\mathbf{a}\,.\,\mathbf{Y})\mathbf{c},$$

from which, by arguments similar to those already used, it follows that $\mathbf{Y}=0$ and that accordingly

$$\lambda_{\mathbf{b}_1,\mathbf{c}} = \lambda_{\mathbf{b}_1+\mathbf{b}_2,\mathbf{c}} = \lambda_{\mathbf{b}_2,\mathbf{c}}.$$

Thus λ is independent of its suffix $\mathbf{b}$. Similarly we can show that it is independent of the remaining suffix $\mathbf{c}$, or, more quickly, notice that since

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = -\,\mathbf{a}\wedge(\mathbf{c}\wedge\mathbf{b}),$$

* Other proofs with a similar object have been given by Neville (*Math. Gazette*, XVII, 320, 1933) and Lowry (*ibid.*, XVIII, 199, 1934). Our attention was drawn to these earlier proofs after this note was written, by the Editor.

we must have by (3)

$$\lambda_{\mathbf{b},\,\mathbf{c}} = \lambda_{\mathbf{c},\,\mathbf{b}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

7. Thus λ is the same for all sets of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and can be evaluated by considering any convenient special set. For this we choose $\mathbf{a} = \mathbf{b} = \mathbf{i}$, $\mathbf{c} = \mathbf{j}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a positive * orthogonal set of unit vectors, for which $\mathbf{i} \wedge \mathbf{j} = \mathbf{k}$, and $\mathbf{k} \wedge \mathbf{i} = \mathbf{j}$. In this case

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{i} \wedge (\mathbf{i} \wedge \mathbf{j}) = \mathbf{i} \wedge \mathbf{k} = -\mathbf{j},$$

and the right-hand side of (2), since $\mathbf{i}^2 = 1$ and $\mathbf{i} \,.\, \mathbf{j} = 0$, is $-\lambda\mathbf{j}$. Hence $\lambda = 1$.

8. The following proof is of an entirely different character. Let $\mathbf{d}$ be any vector, and consider its scalar product with $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$. By (3), this is equal to

$$\lambda_{\mathbf{a};\,\mathbf{b},\,\mathbf{c}}[(\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{d}) - (\mathbf{a} \,.\, \mathbf{b})(\mathbf{c} \,.\, \mathbf{d})]. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

But also, by the known rules of operation of the *scalar* triple product,

$$\begin{aligned} \mathbf{d} \,.\, \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= \mathbf{a} \,.\, (\mathbf{b} \wedge \mathbf{c}) \wedge \mathbf{d} = -\mathbf{a} \,.\, \mathbf{d} \wedge (\mathbf{b} \wedge \mathbf{c}) \\ &= -\mathbf{a} \,.\, \lambda_{\mathbf{d};\,\mathbf{b},\,\mathbf{c}}[(\mathbf{d} \,.\, \mathbf{c})\mathbf{b} - (\mathbf{d} \,.\, \mathbf{b})\mathbf{c}] \\ &= \lambda_{\mathbf{d};\,\mathbf{b},\,\mathbf{c}}[(\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{d}) - (\mathbf{a} \,.\, \mathbf{b})(\mathbf{c} \,.\, \mathbf{d})]. \end{aligned}$$

By comparison with (7) it follows that

$$\lambda_{\mathbf{a};\,\mathbf{b},\,\mathbf{c}} = \lambda_{\mathbf{d};\,\mathbf{b},\,\mathbf{c}},$$

so that λ is independent of the first of the vectors in $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$. We can therefore replace the symbol $\lambda_{\mathbf{a};\,\mathbf{b},\,\mathbf{c}}$ by $\lambda_{\mathbf{b},\,\mathbf{c}}$.

Again, by the same rules of operation,

$$\begin{aligned} \mathbf{d} \,.\, \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= (\mathbf{b} \wedge \mathbf{c}) \,.\, (\mathbf{d} \wedge \mathbf{a}) = \mathbf{b} \,.\, \mathbf{c} \wedge (\mathbf{d} \wedge \mathbf{a}) \\ &= \mathbf{b} \,.\, \lambda_{\mathbf{d},\,\mathbf{a}}[(\mathbf{a} \,.\, \mathbf{c})\mathbf{d} - (\mathbf{c} \,.\, \mathbf{d})\mathbf{a}] \\ &= \lambda_{\mathbf{d},\,\mathbf{a}}[(\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{d}) - (\mathbf{a} \,.\, \mathbf{b})(\mathbf{c} \,.\, \mathbf{d})], \end{aligned}$$

so that also $\lambda_{\mathbf{b},\,\mathbf{c}} = \lambda_{\mathbf{d},\,\mathbf{a}}$.

Hence, since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are independent vectors, λ is independent of the vector suffixes $\mathbf{b}$, $\mathbf{c}$. Hence λ is a constant, and the proof is completed as in § 7.

9. It will be noticed that the proof contained in § 6 depends essentially only on the *additive* properties of vectors. The proof contained in § 8 is shorter, but depends on more advanced properties of vectors, namely on the invariance of the scalar triple product under cyclical interchange of its factors. This property is usually established by showing that the scalar triple product $\mathbf{a} \,.\, \mathbf{b} \wedge \mathbf{c}$ measures the *volume* of the parallelepiped defined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

* It can be shown that all orthogonal triads can be superposed on one or the other of two triads which cannot be so superposed ; one of these is arbitrarily denoted as *positive*, the other as *negative*. *Any* triad can be deformed, by displacement of its rays through angles less than $\frac{1}{2}\pi$, into an orthogonal triad, and is positive or negative according to the sign of the resulting orthogonal triad. The vector product $a \wedge b$ is then defined as such that a, b and $a \wedge b$ form a positive triad. A positive triad may be " right-handed " or " left-handed ".

It therefore depends on appeal to the more advanced concept of *volume*. Once we have established the vector triple product formula by the method of § 6, it should be possible to use this to establish the invariance of the scalar triple product under cyclical interchange of its factors without appeal to the concept of volume. The follow-piece of analysis achieves this.

We have

$$[\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})]^2=[\mathbf{b}(\mathbf{a}\,.\,\mathbf{c})-\mathbf{c}(\mathbf{a}\,.\,\mathbf{b})]^2$$
$$=\mathbf{b}^2(\mathbf{a}\,.\,\mathbf{c})^2+\mathbf{c}^2(\mathbf{a}\,.\,\mathbf{b})^2-2(\mathbf{b}\,.\,\mathbf{c})(\mathbf{c}\,.\,\mathbf{a})(\mathbf{a}\,.\,\mathbf{b}).$$

But since $(\mathbf{p}\wedge\mathbf{q})^2=\mathbf{p}^2\mathbf{q}^2\sin^2\widehat{\mathbf{p}\,\mathbf{q}}=\mathbf{p}^2\mathbf{q}^2(1-\cos^2\widehat{\mathbf{p}\,\mathbf{q}})$
$$=\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\,.\,\mathbf{q})^2,$$

we have also $[\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})]^2=\mathbf{a}^2(\mathbf{b}\wedge\mathbf{c})^2-(\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c})^2$
$$=\mathbf{a}^2[\mathbf{b}^2\mathbf{c}^2-(\mathbf{b}\,.\,\mathbf{c})^2]-(\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c})^2.$$

Equating the two expansions we have

$$(\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c})^2=\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2-\mathbf{a}^2(\mathbf{b}\,.\,\mathbf{c})^2-\mathbf{b}^2(\mathbf{c}\,.\,\mathbf{a})^2-\mathbf{c}^2(\mathbf{a}\,.\,\mathbf{b})^2$$
$$+2(\mathbf{b}\,.\,\mathbf{c})(\mathbf{c}\,.\,\mathbf{a})(\mathbf{a}\,.\,\mathbf{b}).$$

Hence $(\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c})^2$ is unaltered by cyclical interchange of the factors, and hence either

$$\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}=+\mathbf{b}\,.\,\mathbf{c}\wedge\mathbf{a},$$

or

$$\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}=-\mathbf{b}\,.\,\mathbf{c}\wedge\mathbf{a}.$$

But if the latter equality held good, we should have, by its repeated application,

$$\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}=-\mathbf{b}\,.\,\mathbf{c}\wedge\mathbf{a}$$
$$=+\mathbf{c}\,.\,\mathbf{a}\wedge\mathbf{b}$$
$$=-\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c},$$

which is in general a contradiction. Hence $\mathbf{a}\,.\,\mathbf{b}\wedge\mathbf{c}=\mathbf{b}\,.\,\mathbf{c}\wedge\mathbf{a}$, and the invariance of the scalar triple product under cyclical interchange of its factors is established from the vector triple product formula without appeal to the concept of volume.

S. CHAPMAN and E. A. MILNE.

1241. Such differences as may arise (between Great Britain and the United States) when it comes to the practical implementation of policy are accepted as the heritage of history and geography. But they do not prevent the two Powers from moving forward on parallel lines to face a future which threatens the same dangers for both. And the distance between those parallel lines is narrowing. One day it may be imperceptible.—*Sunday Times*, January 2, 1938. [Per Mr. B. M. Neville.]

1242. . . . the bull's a pretty strong thruster; he weighs a ton and more. A ton and more charging at twenty miles an hour comes with an impact of twenty tons.—John Masefield, *Eggs and Baker*, p. 48. (Heinemann, 1936.) [Per Dr. J. McWhan.]

PLACE OF LOGIC IN THE TEACHING OF GEOMETRY.*

By M. Black.

Of all the reasons urged in favour of the teaching of mathematics and the favoured position of mathematics in the secondary school curriculum, the most widespread is the belief that to learn mathematics is valuable as a training in the art of reasoning. In this paper I want to explore the implications of this belief and its bearing upon the teaching of geometry. I shall try to avoid a pronouncement on whether the belief in the value of mathematics as a training in reasoning is justified, by adopting the mathematical method of exploring the consequences of an assumption.

First, I will call three witnesses to support my contention of the pervasiveness of a belief in the value of mathematics as a training in reasoning.

From the statements of many philosophers who have stressed this aspect of mathematics I choose a remark made by Berkeley :

" It hath been of old remarked that Geometry is an excellent logic. And it must be owned that when the definitions are clear ; when the postulata cannot be refused, nor the axioms denied ; when from the distinct contemplation and comparison of figures, their properties are derived by a perpetual, well-connected chain of consequences, the objects being well kept in view and the attention ever fixed upon them ; there is acquired a habit of reasoning, close and exact and methodical ; which habit strengthens and sharpens the mind, and being transferred to other subjects is of general use in the enquiry after truth." (" The Analyst ", 2. *Works* (London, 1798), 3, 10.)

There you have stated, on the authority of one of the world's greatest philosophers and with a classic precision, the doctrine whose implications I want to trace in detail ; the doctrine that the study of mathematics " sharpens the mind " and produces habits which can be " transferred to other subjects ".

I do not think Berkeley ever tried to teach mathematics to groups of small boys of middling intelligence. My next witness is one who speaks with intimate knowledge of practical teaching problems in the United States. In an article written in 1930 Professor Reeve (head of the Mathematics Department in Teachers' College, Columbia University) said : " We must continually repeat and emphasise the fact that the purpose of geometry is to make clear or popular the *meaning of demonstration, the meaning of mathematical precision and the pleasure of discovering absolute truth.*" (W. D. Reeve, " The Teaching of Geometry ", National Council of Teachers of Mathematics, *Year Book*, 1930, page 13.)

Here Berkeley's doctrine is applied specifically to the teaching of geometry, and stated with great emphasis. Professor Reeve insists

* Paper read to the London Branch of the Mathematical Association on Friday, 18th February, 1938.

that the purpose of geometry is " to make clear the meaning of demonstration ".

My last witness is of less public distinction than the two authorities whose remarks I have quoted. A class in a girls' secondary school was asked, a short while ago, why they were learning geometry. After a long pause one girl raised her hand, and said, " Please, sir, geometry strengthens the brain." So this girl, who might serve as a representative of thousands of school children who must have put the same question to themselves, implies almost in the exact words of a great philosopher that the main object of geometry is the formal one of in some way training the mind.

I believe that this doctrine is the most widespread of all justifications for the learning and the teaching of geometry. Not only does it recur continually in the works of mathematicians and philosophers, but it is the reason which rises instinctively to the lips of anyone who is asked how the large place occupied by mathematics in the school curriculum can be justified.

This widespread belief in the value of mathematics in general and geometry in particular as an instrument of mental discipline needs to be contrasted with the experimental work carried out during the last forty years on the whole issue of the " transfer of formal training ". I have neither the time nor the competence to enter, on this occasion, into detailed discussion of this experimental evidence, and would refer you for further information to the article written by Professor Hamley, entitled " Formal Training " (*British Journal of Experimental Psychology*, November, 1936, pp. 233-249).

It is sufficient for our purposes to point out that the pioneer work, undertaken at the turn of the century by Thorndike, tended to prove that the claims made for mathematics, as indeed for other subjects, as an instrument for mental training were at least considerably exaggerated. Careful experiments with parallel groups of children, of which one had and the other had not been exposed to mathematical training, showed that in many cases specific training in mathematical material not only did not improve power to reason about non-mathematical material, but might in certain cases even act in a negative sense. These results, which have been described as a " bombshell in the educational camp ", aroused fierce controversy and have stimulated a long series of research on the same topic. Subsequent work has tended to modify the alarm experienced by teachers and educationalists at the first results of psychological experiment on the problem of transfer. It has been shown that in favourable cases some transfer does occur, and that the problem, in Professor Hamley's words, is not so much " whether transfer of training actually occurs, but *under what conditions* it does occur ". But the detailed and still unconclusive research upon this important subject, for which reference should be made to Professor Hamley's article, does show as clearly as we can expect for any principle of teaching, that if the teaching of mathematics is valuable as a training in clear thinking about non-mathematical material, such train-

ing will occur only under conditions which are specifically designed to produce it. The elements which transfer from mathematics to non-mathematical situations will be the "common elements" in the two situations. We need not take the phrase "common elements" in too narrow a sense; the possibilities of transfer affect not merely mathematical facts which may be common to the secondary curriculum and situations which arise in the ordinary man's normal occupations; elements of method, habits, and above all sentiments and ideals may in favourable cases be transferred. The point that needs emphasis now is that if a method or a sentiment or a habit is to carry over from mathematics to ordinary life, the teaching must bring the common element to be transferred into explicit consciousness. And the amount of useful transfer that can, even in the most favourable circumstances, occur will be limited by the extent to which the method or skill is really common to both situations.

I propose, then, to take the claim of geometry to be a training in reasoning quite seriously, and to examine the subject-matter of secondary school geometry in some detail with a view to deciding which parts of the subject do and which do not particularly lend themselves to the transfer of a skill in reasoning. I wish to restrict my remarks specifically to geometry, because the claim that mathematics can be a training in reasoning, though it is made of all parts of the subject is made most strenuously on behalf of geometry. I shall begin by analysing those aspects of geometrical subject-matter and method which are not particularly favourable to training in the art of reasoning about everyday affairs.

First in the list of geometrical elements which do not lend themselves easily to transfer we shall need to place certain specific geometrical skills, which together constitute a large part of the subject-matter of secondary school geometry. Consider, for example, a simple rider like the following: Suppose a child is asked to prove that, in the given figure (in which $CA = CK$, $CB = CL$, and $\angle ACK = \angle BCL$), $AB = KL$. It would be difficult to find a much simpler

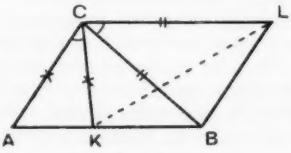

FIG. 1.

exercise in reasoning, and the undoubted difficulty which children often find with simple riders of this sort is almost certainly due to difficulty in dissecting a complex figure into those elements which are relevant to their purpose. Once a child can see the triangles CAB and CKL separately from the figure (as it were in relief), much of the difficulty of the exercise is overcome. So well is this known that experienced teachers systematically draw relevant parts of such

diagrams separately as an aid to the process of dissection required. The process of analysis which such riders demand appears to be specifically bound to geometrical figures. I am not unaware that the process of analysis based on judgments of relevance is to be found in effective thinking concerning every kind of material, but it does seem that the difficulties which a child feels in a geometrical rider are not difficulties of analysis in general, but difficulties of visualising certain geometrical configurations.

Of similar examples of skills which appear to be bound to geometrical subject-matter, one deserves special mention—the ability to visualise continuous transformations of geometrical figures. Consider an example like the following : Suppose, in the course of some geometrical rider, we are presented with the figure shown and know that AM is a tangent to the circle at A and that AL is parallel to

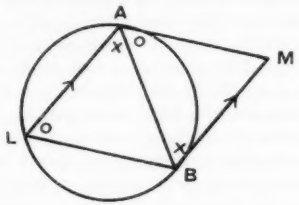

FIG. 2.

MB, and wish to decide whether the two triangles shown are or are not congruent. At first blush they " look " congruent. We at once see that $\angle MAB = \angle ALB$ in the alternate segment, and that on account of the parallelism of MB and AL, $\angle MBA$ = corresponding $\angle BAL$. We now see that, although the two triangles have one side and two angles of the one equal to one side and two angles of the other, they are not necessarily congruent, for the angles are not situated in corresponding positions. Notice that we have not proved that the triangles are not congruent ; all we know is that we do not know whether they are congruent, and the question remains open. Now in such a situation as this the kind of procedure which a child with a flair for geometry may adopt is to argue in the following way. The data consist only of the fact that AM is a tangent at A and that AL is parallel to BM. Very well, then. We will keep AM and AL fixed in position and allow the point B to wander along the arc of the circle towards A. *It is obvious that* (this is the point at which the appeal to visualisation enters) this can be done ; and *we can see* (this is the second appeal to visualisation) that when B is very close to A the triangles BAM and BAL are quite dissimilar in size and shape, and are, therefore, not congruent. Informal arguments of this sort play, as all experienced teachers know, an important, perhaps decisive, part in the learning of geometry. Once we know that the triangles are not congruent we can very often find the proof without difficulty ; the conviction of the

character of the result helps beginners, especially in searching for a proof.

This example has been set out in detail, because it illustrates a mode of thought which permeates the whole of elementary geometry. As in the case of the ability to dissect figures into their parts and to hold these parts distinctly in the imagination, the ability to visualise the possibilities and results of continuous deformation of figures seems to be a skill which is closely bound to the material upon which it is exercised. Speaking subject to the possibility of subsequent correction by psychological experiments, I should be surprised if ability to imagine minute geometrical films were at all closely connected with success in reasoning about everyday situations. Ability to visualise motion would be of use in many technical operations involving the handling of machinery, but the number of children who will be called upon to have extensive dealings with machinery will be a minority of the school population.

The second aspect of the subject-matter of the geometrical curriculum in schools which seems to me unfavourable to the possibility of transfer consists in the large body of geometrical knowledge which the child is ultimately to learn as remembered facts. In theory we can dispense with memory in geometry and prove each result as we need it, but any child who attempted to do so would be either a fool or a genius. Success in learning geometry depends largely upon the possession in a readily accessible form of a large body of geometrical facts, from the various conditions of congruence to the A and B formulae in trigonometry. These facts in themselves may be interesting, and in so far as they contribute to the learning of geometry may be of help to the general plan of promoting reasoning in everyday situations, but in themselves they have no direct or obvious application to such situations. Knowledge of the concurrency of medians is not a conspicuous help in making ends meet on pay day, and a knowledge of the properties of the ellipse is of no great assistance in thinking clearly about the elliptical utterances of politicians. I conclude, therefore, that the large portion of geometrical instruction which is and must be devoted to the assimilation of geometrical facts does not lend itself easily to improvement in reasoning about everyday life.

The next objection to the use of geometry as a training in clear thinking is even more serious than the objections so far discussed that have been based upon the specific character of geometrical skills and geometrical knowledge. For it is based upon the character of mathematical reasoning itself. The fact that in mathematics there is no intermediate position between truth and falsehood, and that we are not encouraged to argue in terms of probabilities, has one important consequence. Where we are concerned with an argument involving probabilities the falsity of the conclusion does not invalidate the process of argument, even if the premises are true. If I want to meet a friend at a station and decide that if he is coming to town that day he will almost certainly travel by the

4 o'clock, I may be arguing in a perfectly sensible and valid fashion, even although I wait for him at the station in vain, and discover subsequently that on the day in question he came to town in a motor car. The argument on which my action was based, namely, that he would very probably be at the station at a certain time, was a valid argument, and the fact that he was not, the fact that my expectations were not verified, is not a criticism of the argument itself. In mathematics the situation is different: an expectation, *e.g.* of the numerical value of a variable, which is not fulfilled invalidates the argument. Conversely, when we obtain a mathematical result we believe it, at the elementary level which we are discussing, with absolute certainty. We can express this shortly by saying that mathematics deals in certainties and commonsense in probabilities. Contrast the proof of Pythagoras' theorem with the process of argument in some everyday situation, *e.g.* a train of thought started by seeing a newspaper poster with the words "Great Revolt reported in Abyssinia". In the geometrical proof of Pythagoras we are dealing with and demand a certainty independent of observation; we need perform no measurement and our conclusions are uninfected by doubt. Contrast this with the argument about the poster. My only action upon seeing the poster in question was to say to myself: "The word 'reported' means that this is just a rumour. If the headline had run 'official', there might have been one chance in ten, say, that the news was reliable. As it is, I expect the 'Morning Banner' is rather short of news to-day." Now I believe this to be a comparatively intelligent train of argument, and to be a specimen of the type of clear thinking which we should at all events not wish to discourage in our children. But it is to be noticed that the whole force of the argument rests upon an interpretation of the significance of the word 'reported', *i.e.* upon the isolation of a significant element on the basis of previous knowledge. It is a good example of the informal methods which commonsense adopts. I do not believe that trains of argument in geometry fall into the same pattern.

I do not wish to deny that there is a place for commonsense in the learning and teaching of mathematics. Children who assume from the tone of the teacher's voice, as so many of them constantly do, the probable conclusion as to the kind of answer expected of them, are arguing in a commonsense way based upon knowledge of the behaviour of the schoolmaster; the same is true of the child taking an examination who, having written out a theorem for the first part of the question, proceeds to argue that the rider in the second part probably depends upon that theorem. Intelligent learning of mathematics and of any other subject clearly involves a constant use of commonsense modes of thought, but the subject-matter of mathematics is not itself a commonsense pattern. This is, in my opinion, a rather serious obstacle to the value of mathematics as a training for clear thinking in ordinary life.

My next objection to the possibility of using geometry for pur-

poses of transfer to reasoning in everyday situations is closely connected with the last. Mathematics both in subject-matter and process is characteristically neat and tidy (I cannot think of better words for what I mean) in a fashion which is not paralleled in ordinary life. Everything superfluous for the argument has been expelled before the subject-matter ever gets to the schoolroom ; and the whole tendency of the pure mathematician is to accentuate this tendency. To take an example which is of some interest because it shows at least one case where teaching of mathematics, which is effective in the short run, tends to make transfer difficult. I refer to the fact that the data of a problem that children are asked to solve are almost invariably both necessary and sufficient. The kind of advice which teachers of mathematics will give, and rightly, is : " Look at the question. Have you used all the data? Go back and try to make some use of any piece of information which you have not already brought into the solution." How unlike a situation in everyday experience, where half the trouble is that we never know whether the data is sufficient or redundant.

The last obstacle which the subject-matter of geometry presents to the transfer of the power to reason about ordinary situations is its extremely systematic character. The 1923 geometry report of the Mathematical Association recommended that a particular stage in the teaching of the subject should be devoted to making explicit the interconnection of the various theorems. I doubt whether many schools find time for teaching this very systematically, but the fact that all theorems hang together in one connected structure is constantly brought to the attention of the learner throughout his exploration of elementary geometry. Here we have a situation unlike anything that the child is to meet in later life. Outside the legal system or a highly developed science like physics, the situations concerning which he will be called upon to reason will be fragmentary and not systematic, disassociated and not organised. A too close linkage between the salutary exercise of argument and a systematic body of theorems may tend to discourage the exercise of reasoning in situations such as those of politics, which do not present the tidy, articulated character dear to the heart of the pure mathematician.

These are formidable obstacles, for it is hard to see how the aspects of geometry which I have mentioned could, even if we should want to, be removed from the geometrical curriculum. For they are essential aspects of the nature of geometry, and if we want to use geometry as a training for clear thinking it is geometry that we must teach.

What can we say in defence of geometry as a training in clear thinking? When all that can be urged against the use of geometry as a training in reasoning has been said, it still remains necessary to emphasise that of all school subjects mathematics is the one which makes the most constant demands upon argument. Moreover, this is the one subject in the curriculum where rational argument can

be shown in action and can be shown to be successful. In all subjects the type of reasoning which I have called commonsense argument will have its place, but the verification of the results will in general be impracticable. In history, for example, the child must take his facts largely on trust, on the authority of the teacher. In the nature of the case the child cannot have direct access to events which are past and gone ; nor is he in a condition to evaluate the complex, indirect evidence on which historical research is based. In the most favourable case children may get some confirmation of broad facts like the Roman invasion, by handling coins or inspecting ancient monuments, but the contact with the facts which verify arguments is infrequent and unconvincing. It might be said that the process of verification on which I am laying stress belongs more properly to lessons in natural science, but so much of these is rightly occupied with the acquisition of useful information that the opportunities for verification are small. The kind of experiment which the verification of scientific theories demands is too complex to produce conviction with the average child. Most children's attitude towards experiments performed in the scientific laboratory is a cynical one. Experiments in physics especially are regarded as results to " cook ". Science at the elementary levels is concerned much more with observation than with verification. In geometry, on the other hand, it is because we are concerned with the very simplest and most primitive kind of physics, with objects which any child could reproduce at will, by using a few sheets of cardboard and a bottle of gum or a piece of slate and a lump of chalk, that verification is easy and decisive. I believe that a large part of the value of the teaching of mathematics depends upon the extent to which we can arouse and feed the demand for verification.

I should like to give an example of the " thrill of verification " from a lesson which I observed last year. The teacher was discussing with an elementary class in their first year of geometry the number of diagonals which could be obtained in a plane polygon of a given number of sides. The class were drawing four-sided, five-sided, six-sided polygons, and constructing a table showing the connection between the number of sides and the number of diagonals. After a while the construction of the figures and the counting of the number of diagonals became increasingly difficult, and at the " psychological moment " the teacher asked how many diagonals a hundred-sided polygon would have. There were protests from the weak members of the class, and a look of helplessness on the face of the rest. The class then proceeded to the simple discussion which shows that the number of diagonals in an n-sided polygon will be $\frac{1}{2}n(n-3)$, and proceeded first to calculate and then to verify by drawing, the value of the answer for $n=9$, 10, and so on. They also worked out the answer for $n=100$, " for fun ". This seemed to me an excellent lesson, because interest was obviously aroused and because the excitement shown when it came to drawing the nine-, ten-sided figures seemed to me to be an example of the " thrill of

verification " of which I spoke. I might add that in a few days' time some members of the class produced with an air of triumph elaborate diagrams for $n = 100$, and proudly announced that the result they had calculated in class did in fact give the right number of diagonals.

It seems to me that we have here on a small scale the very same impulse at work which makes us take inordinate pleasure in Leverrier's discovery of Neptune by calculation, which will always remain one of the greatest triumphs of mathematical method.

But the value of geometry as stimulating the belief in the possibilities and the value of rigorous thinking depends entirely upon the success with which children are able to learn geometry. If children are to be inspired with the possibilities of human intelligence, it is essential that they should believe themselves to be intelligent, and to be able to use their intelligence successfully. Here, again, my hopes for the value of teaching geometry are qualified by the fact that many children find so much difficulty with the subject. To the extent to which they do find serious difficulty and become discouraged, the possibility of using geometry as a field for stimulating belief in the value of intelligence is reduced almost to vanishing point.

When we come to consider the techniques which will best promote the transfer of mathematical skill to non-mathematical situations, there will be one all-important guiding principle to consider. If we want transfer to occur we must see that ample exercise is given in the application of mathematical principles to the kind of subject-matter to which we wish transfer to be made. If we want children to apply the principles of reasoning with which they have to deal in geometry to situations in which they will be called upon to argue about politicians or employers, it is essential that we should give them exercise in arguing about the politicians and employers, as well as about points, lines and surfaces. How far we are from any such technique can be seen by examining any of the constant succession of the new geometrical text-books which issue from the publishers. We shall find that their vocabulary is almost entirely geometrical ; a child might be pardoned for thinking that geometry has nothing to do with the furniture of everyday life. To remedy this we shall need to take special trouble to illustrate each and every geometrical principle by the use of non-geometrical material. Many teachers will, I know, already enliven the academic atmosphere of a mathematics lesson by making occasional reference to the outside world. Admirable as such excursions into reality are, our programme demands a much more systematic and intensive use of concrete material. The reference to non-mathematical subject-matter must appear not as an embellishment or a relief, but as an integral part of the lesson itself. A class must be made to feel that one of the prime objects of learning geometry is to get to know intelligent ways of thinking about every kind of material. Those who feel that such a programme is impracticable might be invited to consider

the extreme generality of mathematics itself. From its very nature mathematics is applicable to a far wider range of experience than is commonly emphasised in the teaching of the subject. Let us take a simple example of the operation of combining like terms in elementary algebra. Taught *in vacuo* without any reference to the purpose which determines the value of this operation, the topic is bound to appear to the child as one more of the meaningless tricks with which the syllabus, from his point of view, is cluttered. Suppose, however, we draw the attention of the child to anybody, say, a cashier at a cinema, who is to perform lots of arithmetical calculations of the same type. If she is wise the cashier adding up her takings will not work out five 9d.'s, then three 1/3d.'s, then another seven 9d.'s, then another four 1/3d.'s, and so on, but will inevitably do all the 9d. sums in one operation, and then all the 1/3d.'s in another operation, and add the results. Similarly, any office boy addressing circulars is likely to receive the sack if he writes the address on the envelope, inserts the circular and sticks the flap for each envelope in turn. Having probably received an expensive secretarial training at Pitman's, he will remember the principle of organisation which insists upon performing all operations of the same type before proceeding to a second operation. It is more economical to write all the addresses before proceeding to insert all the circulars. Here we have a simple example of how the principle of like terms can be made to appear as a special case of the much more general and sensible principle of organisation of labour.

The method to be adopted in geometry is largely the same. In introducing indirect proof, *e.g.* to children, much can be done to allay any feeling of suspicion by pointing out with numerous examples taken not from geometry, but from everyday life, that indirect proof is an argument of frequent occurrence. If from an outside window I see a postman approaching the house and pause half-way down the stairs disappointed because I have heard his knock at the next house, I am employing an argument which is indirect in form. "If the postman were bringing letters to my house he would have knocked here before knocking at the next house. But this is not the case. Therefore there are no letters for me." Similar exercises (on such topics as definitions, axioms, assumptions, etc.) will go far to promote appreciation of the modes of proof involved in geometrical thinking, and will promote their ready application to everyday life. M. BLACK.

1243. And Sight, kept clear, becomes curiously, Insight. A thousand had seen apples fall before Newton. But Newton was dowered with the Spirit of the Open Road.—David Grayson, *Adventures in Friendship*, p. 49. [Per Mr. W. J. Langford.]

THE INCOMPLETENESS OF "COMPLETE" PRIMITIVES OF DIFFERENTIAL EQUATIONS.

By H. T. H. Piaggio.

1. Dr. C. N. Srinivasiengar, in his published papers (7, 8, 9 and 10), and Dr. W. E. Deming, in a private letter, gave examples of differential equations for which the so-called complete primitives are not really complete, and yet the supplementary solutions are not singular solutions in the envelope sense. In the ensuing correspondence I expanded a remark of E. B. Wilson (11): "It may happen that the arbitrary constant C enters into the expression $F(x, y, C)=0$ in such a way that when C becomes positively infinite (or negatively infinite) the curve $F(x, y, C)=0$ approaches a definite limiting position which is a solution of the differential equation; such solutions are called *infinite solutions*." Further consideration of the subject gradually led from some results which are almost obvious to others (especially those in italics in § 4) which are believed to be new.

2. As a starting point we notice that given a differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

in which the left-hand side of the equation is identically equal to $v(x, y)\,.\,du(x, y)$, then in addition to the complete primitive $u(x, y)=c$, there is also the solution $v(x, y)=0$. This is mentioned in advanced treatises, *e.g.* those of Ince (4) and de la Vallée Poussin (6), but very little is said about the relation between the complete primitive and the supplementary solution. We shall begin the discussion of this relation informally, by the aid of examples, and then enunciate a general theorem.

As a first example consider the differential equation

$$\frac{dy}{dx} = 1 - y^2.$$

The usual method of integration is to separate the variables, giving

$$dx = \frac{dy}{1-y^2} = \tfrac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy,$$

whence

$$x + c = \tfrac{1}{2} \log \frac{1+y}{1-y},$$

and $y = \tanh(x+c)$, the complete primitive. But the separation of the variables involves division by the factor $(1-y^2)$, which is illegitimate if this factor is zero. Hence the final result excludes the possibility that $y = \pm 1$, which gives two solutions not included in the complete primitive. It might be supposed that these are singular solutions, but if we define a singular solution as a solution representing the envelope of the *characteristics* (a term which we shall use to denote the family of curves represented by the complete

primitive), then these two extra solutions are not singular. They represent the common asymptotes of the characteristics, which have no envelope. It will be remembered that an envelope is defined as a curve which touches every member of a family of curves, and which, at every point of itself, is touched by a member of the family. The second part of this definition excludes a common asymptote, which is touched by every member of the family at only one point, namely, that at infinity. Thus the two extra solutions are of the same nature as those included in the " complete " primitive, which from a strictly logical point of view would be more fittingly called *incomplete*. However, given $y=\tanh(x+c)$, we can easily derive $y=\pm 1$ by taking $c=\pm\infty$. This means that the upper asymptote is very close indeed to the part of $y=\tanh(x+c)$ near the origin, when c is very large and positive, with a corresponding result for the lower asymptote. Thus in this example the residual primitive is made up of two limiting forms of the complete primitive.

It should be noticed that an apparently trivial change in the form of the arbitrary constant may make a limiting form appear to be included in the complete primitive. Thus if in the integration above we replace c by $\frac{1}{2}\log k$, we get $(1+y)/(1-y)=ke^{2x}$, and the solution $y=-1$ is given by $k=0$. But $k=0$ is equivalent to $c=-\infty$, so there is just the same logical difficulty in the second form as in the first ; it is not removed, but merely hidden.

A less obvious example is $y=\tan(x+c)$, derived from the differential equation $\dfrac{dy}{dx}=1+y^2$. The residual primitive is $1+y^2=0$. At first sight one might think that the characteristics had no common asymptotes, but this is not so. The asymptotes exist, as before, only this time they are imaginary, and they are derived from imaginary characteristics. To obtain $y=\pm i$ as a limiting form of $y=\tan(x+c)$, we replace c by ik, where k is real, and then take $k=\pm\infty$. This procedure is quite legitimate. The primitives of differential equations lose their generality unless the arbitrary constants are allowed to take all values, real, imaginary, and complex.

A third example arises from the differential equation

$$\left(\frac{dy}{dx}\right)^2=y^3-y^2,$$

with a complete primitive $y=\sec^2\frac{1}{2}(x+c)$ and the residual primitive $y=0$. The residual primitive is now a *real* line that is a common asymptote of imaginary characteristics. It can be derived by replacing c by ik, and then taking $k=\pm\infty$. There is also a singular solution, $y=1$, representing the envelope. This example has been the subject of some controversy. The p-discriminant is y^2-y^3, and if, as many writers do, we *define* a singular solution as a solution which causes the p-discriminant to vanish, then $y=0$ will be regarded as a singular solution. In the first two differential equations, which are of the first order, no p-discriminants exist, so the residual

primitives cannot possibly be derived from them. As the residual primitives are of the same nature in all three examples, representing common asymptotes, the envelope definition of a singular solution seems preferable. The other definition achieves simplicity of statement by ignoring essential geometrical differences.

This example illustrates another important point. The differential equation may be written in the form

$$\frac{dy}{dx} = \pm y(y-1)^{\frac{1}{2}}.$$

The factor $(y-1)^{\frac{1}{2}}$, which gives rise to a singular solution, occurs with an index lying between 0 and 1, while the factor y, giving rise to a residual primitive, occurs with an index 1. In the simpler example $\frac{dy}{dx} = y^n$, where $n > 0$, we get

$$(1-n)(x+c) = y^{1-n}, \quad \text{if} \quad n \neq 1,$$

and

$$x + c = \log y, \quad \text{if} \quad n = 1.$$

$y = 0$ is always an integral, but it is a residual primitive, derived by taking $c = -\infty$, if $n \geqslant 1$, whereas it is a singular solution (in the envelope sense) if $n < 1$. This suggests a corresponding theorem for $\frac{dy}{dx} = g(y)$. If $g(a)$ vanishes in consequence of $g(y)$ having a factor $(y-a)^n$, then $y = a$ is a residual primitive if $n \geqslant 1$, but a singular solution if $n < 1$.

3. There is no need for the asymptotes represented by residual primitives to be linear. Let us transform the first example

$$\frac{dy}{dx} = 1 - y^2,$$

replacing y by $y - x^2$. This gives $\frac{dy}{dx} = 1 + 2x - x^4 + 2x^2 y - y^2$, with the complete primitive $y = x^2 + \tanh(x + c)$. The residual primitive $y = x^2 \pm 1$ represents the common *parabolic* asymptotes of the characteristics.

In general, we may replace y by $y - f(x)$, and formulate the conditions that this should give a residual primitive or singular solution of the differential equation $\frac{dy}{dx} = g(x, y)$. However, it is better to deal with the more general equation

$$P(x, y)\,dx + Q(x, y)\,dy \equiv v(x, y) \,.\, du(x, y) = 0.$$

The solution $v(x, y) = 0$ may be of various kinds. Firstly, let one or both of the partial derivatives u_x, u_y become infinite when $v = 0$, although u itself and also $vu_x\,(= P)$ and $vu_y\,(= Q)$ all remain finite. $v = 0$ will then be a singular solution, representing an envelope. For example, $dx + \{1 + (x+y)^{\frac{1}{2}}\}dy = 0$ may be written in the form $(x+y)^{\frac{1}{2}} \,.\, d\{y + 2(x+y)^{\frac{1}{2}}\} = 0$, giving the complete primitive

$$y + 2(x+y)^{\frac{1}{2}} = c,$$

and the singular solution $x+y=0$. The characteristics are incomplete parabolas, each of which ends abruptly at the point where it touches the envelope, since $(x+y)$ cannot be negative if $(x+y)^{\frac{1}{2}}$ is to be real. The missing portions are given by $y-2(x+y)^{\frac{1}{2}}=c$. However, there is no abrupt ending of the characteristics of the very similar differential equation

$$dx+\{1+(x+y)^{\frac{1}{3}}\}dy=0,$$

which gives $y+\frac{3}{2}(x+y)^{\frac{2}{3}}=c$, the complete primitive, and $x+y=0$, the singular solution. In the case of $x\,dx+\{y+(x^2+y^2)^{\frac{1}{2}}\}dy=0$, the singular solution $x^2+y^2=0$ should be regarded as representing two imaginary envelopes, $y=\pm ix$, of the incomplete parabolas

$$y+(x^2+y^2)^{\frac{1}{2}}=c,$$

rather than merely the isolated point (0, 0).

Secondly, let $v=0$ make u itself infinite as well as one or both of u_x and u_y, while vu_x and vu_y remain finite. $v=0$ will then give a limiting form of the complete primitive, corresponding to an infinite value of the arbitrary constant.

For example,

$$dx+\{1+(x+y)^{\frac{2}{3}}\}dy\equiv(x+y)^{\frac{2}{3}}\,.\,d\{y-2(x+y)^{-\frac{1}{2}}\}=0$$

gives $y-2(x+y)^{-\frac{1}{2}}=c$, the complete primitive, and $x+y=0$, representing the common asymptote of the characteristics.

Thirdly, u, u_x, u_y may all be finite when $v=0$. In this case the solution $v=0$ makes both P and Q vanish, and it has no necessary geometrical relation of any kind to the characteristics. Such a solution may be called *trivial*. For example,

$$(x+y)^{\frac{1}{2}}dx-2(x+y)^{\frac{1}{2}}y\,dy=0$$

gives the complete primitive $x-y^2=c$, and the trivial solution $x+y=0$. It is best to regard this differential equation as resulting from the union of the simpler differential equation $dx-2y\,dy=0$ with the algebraic equation $(x+y)^{\frac{1}{2}}=0$. Trivial solutions are the unwanted offspring of such unions of incompatible partners. We shall now formulate conditions which will get rid of such solutions and will enable us to distinguish between singular solutions and limiting forms. It is convenient to consider not the equation $v(x, y)=0$, but the simpler equation $y=f(x)$, which is *one* of the consequences of $v=0$. The other consequences can be considered separately. In what follows we shall consider only the case when $f(x)$ is real, as the theorem is in a form applicable only when all imaginary numbers are excluded.

Let $P(x, y)$ and $Q(x, y)$ be single-valued, finite and continuous (possibly one-sidedly, as for square roots and other functions which become imaginary for negative values of the argument) in the domain considered, and be of constant sign in the immediate neighbourhood (possibly one-sided) of $w\equiv y-f(x)=0$. P and Q must

not both be zero when $w=0$. (If this is not satisfied originally, it can always be effected by division by a suitable factor.) *Then if $F(x, w)$ denotes the result of replacing y by $w+f(x)$ in $P+Qf'(x)$, the necessary and sufficient conditions that $w=0$ should be a singular solution are that $F(x, 0)=0$ and $\int_0^w \frac{dw}{F(x, w)}$ should be convergent at its lower limit for all values of x in the domain considered. If only the first condition is satisfied, the integral $w=0$ is a particular integral, obtainable from the complete primitive by giving the constant a particular value (possibly infinity).* This theorem can be traced back to Poisson, who gave a proof for a restricted case. Boole (1) generalised this, but his work lacked some of the refinements demanded by modern analysis. I propose to give these in a future paper. The work is closely analogous to that of my paper " Special Integrals of Lagrange's linear partial differential equation " (5).

Applying this theorem to the second example of this section,

$$P+Qf'(x)=1-\{1+(x+y)^{\frac{1}{2}}\}=-(x+y)^{\frac{1}{2}}=-w^{\frac{1}{2}}=F(x, w). \quad F(x, 0)=0$$

and $\int_0^w \frac{dw}{F(x, w)}=-2w^{\frac{1}{2}}$, so is convergent, hence $w=0$ gives a singular solution.

Similarly in the third example $F(x, w)=-w^{\frac{1}{3}}$, giving again a convergent integral and a singular solution. However, in the fifth example $F(x, w)=-w^{\frac{3}{2}}$, and the integral is divergent, so $w=0$ is a particular integral. As the theorem has been given in a form applicable only to real variables, it cannot be applied to the fourth example.

To show the power of this method, we may apply it to the example $\frac{dy}{dx}=qe^{-1/y}$, where q is a function of x and y which is neither zero nor infinite when $y=0$. Hill (3) said that he did not know of any method of deciding the question whether $y=0$ was a singular solution or not. But if we define the value of $e^{-1/y}$ for $y=0$, which is really indeterminate, as 0, so as to make the function one-sidedly continuous (for positive values of y), and if we impose conditions on q similar to those laid down for P and Q, then we can apply our theorem. The integral $\int_0^y \frac{dy}{q(x, y)e^{-1/y}}$ is divergent, so $y=0$ is a particular integral and not a singular solution.

It will be noticed that the conditions for a singular solution make it necessary, but not sufficient, that at least one of the coefficients P and Q should involve a function with a singularity, *e.g.* $(x+y)^{\frac{1}{2}}$.

4. The integrable " total " differential equation

$$P(x, y, z)dx+Q(x, y, z)dy+R(x, y, z)dz=0$$

can be treated on similar lines. If the left-hand side is identically equal to $v(x, y, z) \cdot du(x, y, z)=0$, then in addition to the complete

primitive $u=c$, there is also the solution $v=0$, which is either a singular solution (in the envelope sense) or a limiting form. *If we impose conditions on P, Q, R similar to those for P and Q in § 3, and denote by $F(x, y, w)$ and $G(x, y, w)$ the results of replacing z by $w+f(x, y)$ in $P+Rf_x$ and $Q+Rf_y$ respectively, then the necessary and sufficient conditions for $w \equiv z-f(x, y)$ to be a singular solution are that $F(x, y, 0)=0=G(x, y, 0)$ and that at least one of the integrals*

$$\int_0^w \frac{dw}{F(x, y, w)} \quad \text{and} \quad \int_0^w \frac{dw}{G(x, y, w)}$$

should be convergent at its lower limit for all values of x and y in the domain considered. If $F(x, y, 0)=0=G(x, y, 0)$ but the convergence condition is not satisfied, then $w=0$ is a particular integral. As before, we deal only with real variables.

The proof is reserved for a future paper, but we may illustrate the theorem by some examples.

$$dx+\{1+(z-x-y)^{\frac{1}{2}}\}dy-dz \equiv (z-x-y)^{\frac{1}{2}} \,.\, d\{y-2(z-x-y)^{\frac{1}{2}}\}=0$$

has the complete primitive $y-2(z-x-y)^{\frac{1}{2}}=c$. $z-x-y=0$ is a singular solution, representing the envelope of the family of surfaces represented by the complete primitive. Here $F(x, y, w)=0$ and $G(x, y, w)=w^{\frac{1}{2}}$, so the second integral is convergent. On the other hand, for

$$z^{\frac{3}{2}}(dx+2ydy)+dz \equiv z^{\frac{3}{2}} \,.\, d(x+y^2-2z^{-\frac{1}{2}})=0,$$

$z=0$ is a limiting form of $x+y^2-2z^{-\frac{1}{2}}=c$. Here $F(x, y, w)=w^{\frac{3}{2}}$ and $G(x, y, w)=2yw^{\frac{3}{2}}$, so both integrals are divergent. Similar results hold for integrable total differential equations in any number of independent variables.

We now turn to " non-integrable " equations of the form

$$Pdx+Qdy+Rdz=0,$$

i.e. those that do not possess a complete primitive involving an arbitrary constant. In this case

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

is not identically zero. Denote it by $v(x, y, z)$. If $v=0$ satisfies the differential equation, it is a singular solution. Let us consider two examples. For

$$zdx+z^2dy+dz=0, \quad v=z^2.$$

As $z=0$ satisfies the differential equation, it is a singular solution. But for

$$ydx+zdy+xdz=0, \quad v=y+z+x.$$

$x+y+z=0$ does not satisfy the differential equation, so it is not a singular solution.

Goursat (2) and de la Vallée Poussin (6) state that v is necessarily

zero for a singular solution of a non-integrable total differential equation. This statement is true only if their severe restrictions on P, Q, R are maintained. If P, Q, R are allowed to have infinite partial derivates, a singular solution may exist which does not make v zero. For example, $z=0$ is a singular solution of

$$z^{\frac{1}{2}}dx + z^{\frac{1}{3}}dy + 2dz = 0,$$

but v comes to $-\frac{1}{6}z^{-1/6}$, and $z=0$ makes this infinite instead of zero. Such solutions have not (as far as I am aware) been noticed before. I hope to give a general treatment later.

It will be noticed that there is a curious difference in the way that singular solutions arise for integrable and non-integrable equations. For the former, it is necessary that at least one of the coefficients P, Q, R should contain a function with a singularity, but this is not so for the latter. The non-integrable equation represents a doubly infinite family of curves orthogonal to the curves of the family $dx/P = dy/Q = dz/R$, but there is no family of *surfaces* orthogonal to this second family of curves. The first family of curves all touch the surface represented by the singular solution, if such exists.

5. Turning now to ordinary differential equations of the second order, with complete primitives containing two arbitrary constants, we have only to let one arbitrary constant become infinite to obtain, in certain cases, a residual primitive with one arbitrary constant. For example, the differential equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$$

gives the complete primitive $x = a + y + b\,\log(y-b)$, and $a/b = +\infty$ gives the residual primitive $y=b$, representing, for each value of b, a common asymptote of the sub-family for which b has that constant value while a varies.

Sometimes we let both constants become infinite, with a certain relation between them. Thus the differential equation

$$8\frac{d^2y}{dx^2} + 9\left(\frac{dy}{dx}\right)^4 = 0$$

gives the complete primitive $(y+b)^3 = (x+a)^2$, representing a doubly infinite set of semi-cubical parabolas, with the tangents at the cusps parallel to the axis of y. If we replace b by $(a^{\frac{3}{2}} - k)$ we get

$$(y-k)^3 + 3a^{\frac{3}{2}}(y-k)^2 + 3a^{\frac{3}{2}}(y-k) = x^2 + 2xa.$$

Dividing by $a^{\frac{3}{2}}$ and then letting a become infinite, we get $y=k$. For each value of k this is a cusp-locus of a sub-family of the semi-cubical parabolas, but the cusps come into the locus only by accident. The locus arises as a limiting form to which very distant members of a certain sub-family, those with vertices $(-a, -a^{\frac{3}{2}} + k)$, approximate in their parts remote from infinity. The locus is

neither an osculating envelope (the geometrical representation of a singular solution of a differential equation of the second order) nor a common asymptote in the ordinary sense of the term.

Similarly if m and n are any two positive integers prime to each other, with $n>m$, then the differential equation

$$m^n\left(\frac{d^2y}{dx^2}\right)^{n-m} - n^m(m-n)^{n-m}\left(\frac{dy}{dx}\right)^{2n-m} = 0$$

has the complete primitive $(y+b)^n = (x+a)^m$, with a residual primitive $y=k$, deducible by replacing b by $\left(a^{\frac{m}{n}} - k\right)$, dividing by $a^{(mn-m)/n}$ and letting a become infinite. The geometrical interpretation is as before, except that the vertices lying on $y=k$ need not be cusps. They may be points of inflexion or of undulation, or even (for $n=2$, $m=1$) ordinary points. In the case of undulation, it might be supposed that the locus is an osculating envelope; this is certainly not so, as the locus, instead of touching the curves, cuts them at right angles.

6. Similar methods can be applied to partial differential equations. Take, for example, $z^2(p^2z^2+q^2)=1$, where

$$p \equiv \partial z/\partial x, \quad q \equiv \partial z/\partial y.$$

The usual method of solution is to assume z to be a function of $x+ay$, $=u$, say.

This gives $$z^2\left(\frac{dz}{du}\right)^2(z^2+a^2)=1.$$

Separating the variables we get $u+b = \pm\frac{1}{3}(z^2+a^2)^{\frac{3}{2}}$, giving the complete integral $(z^2+a^2)^3 = 9(x+ay+b)^2$. Imitating the procedure of § 5, replace b by $\frac{1}{3}a^3 - ak$, divide by $3a^4$, and then let a become infinite. This gives $z^2 = 2(y-k)$, which certainly satisfies the differential equation. It might be supposed that this is a particular case of the "general" integral, representing the envelope of the singly infinite sub-family

$$(z^2+a^2)^3 = 9(x+ay+\tfrac{1}{3}a^3 - ak)^2.$$

But the direction-cosines of the normal at a point on this surface are in the ratio

$$18(x+ay+\tfrac{1}{3}a^2-ak) : 18a(x+ay+\tfrac{1}{3}a^2-ak) : -6z(z^2+a^2)^2,$$

whereas those at a point on $z^2=2(y-k)$ are in the ratio

$$0 : 2 : -2z.$$

These two sets of ratios cannot be the same for any value of a except ∞, so $z^2=2(y-k)$ does not represent an envelope. It may be called a residual integral.

The reader may suspect that the logical flaw in the derivation of the "complete" integral lies, as before, in the separation of the variables. But, like a conjuror who induces his audience to

concentrate their gaze on one spot while he is palming the ace at another, we have now made the false assumption at quite a different part of the argument, namely, where we assume that z is a function of $x+ay$. This is not legitimate when z is a function of y alone, and it is this exceptional case that gives the residual integral. Similar considerations apply to any partial differential equation of the form

$$F(z, p, q)=0.$$

The exceptional integrals of Lagrange's linear partial differential equation will not be discussed here, not because there is too little to say about them, but because there is too much. The analytical aspect has been treated in a paper already mentioned (5), and I hope to deal with the geometrical aspect in another paper.

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University College, Nottingham. H. T. H. Piaggio.

1244. One of the reasons why the Greeks and Romans, who were quite as intelligent as we are, failed to devise more interesting machinery, was to be found in the widespread existence of slavery. Why should a great mathematician waste his time upon wires and pulleys and cogs and fill the air with noise and smoke when he could go to the market-place and buy all the slaves he needed at very small expense?—Hendrik van Loon, *The Story of Mankind*, p. 403. (Per Mr. W. J. Langford.]

1245. To begin with, the earth is not a planet although the astronomers say it is. The planet idea is based upon the whirling globe, and that is simply theory, impossible to prove.

The earth is the largest and most important body in the universe, and all the heavenly bodies—sun, moon, and stars—were made to serve it. That is not only Scripture, but demonstrable science. On the other hand, the whirling globe idea is simply theory, and none know that better than the astronomers as I stated in my letter in your issue of May 14.—Unorthodox, *Weekly Scotsman*, May 21, 1938.

CORRESPONDENCES IN PROJECTIVE GEOMETRY.

By J. A. Todd.

A familiar method of proving theorems in elementary projective geometry is to establish by a geometrical construction the existence of a (1, 1) correspondence between the elements of two simply infinite rational systems (such as ranges of points, or pencils of lines) and to conclude that such a correspondence is projective and to use this result to deduce the theorem desired. The argument underlying this reasoning is that the relation between the coordinates x, y of corresponding elements (x), (y) of the two systems is given by an algebraic equation $f(x, y) = 0$, and that if this relation is such that to each value of x corresponds one value of y and vice versa, then the relation $f(x, y)$ must reduce to a bilinear equation, of the form $Axy + Bx + Cy + D = 0$, which is easily shown to define a projectivity between the corresponding elements (x), (y).

That the relation $f(x, y)$ does in fact so reduce is apparently regarded as obvious by the writers of current English textbooks on projective geometry. In reality it is a theorem of algebra requiring proof. The aim of this note is to furnish such a proof, of sufficiently elementary a nature as to be within the scope of students who are making their first acquaintance with projective geometry: it assumes a minimum of algebraic theory. I also give a proof of the corresponding result for (α, β) correspondences; here, however, I have had to assume rather more. I hesitate to claim originality for the method used though I have not seen it elsewhere, but I feel that the neglect of the subject by English textbooks justifies the present note.

Suppose that $f(x, y) = 0$ is an algebraic equation (which we can clearly suppose without loss of generality to be a polynomial equation) between x and y, with the property that for each value x_1 of x the equation $f(x_1, y) = 0$ determines one and only one value y_1 of y, and vice versa. (It is assumed that the usual convention as to infinite roots is made; this is essential since a projective coordinate may take the "value" ∞.) Let us arrange $f(x, y)$ as a polynomial in y, of degree m (written with binomial coefficients)

$$f(x, y) \equiv y^m a_0(x) + m y^{m-1} a_1(x) + \ldots + a_m(x)$$

where $a_0(x) \not\equiv 0$, and $a_0(x)$, $a_1(x)$, $\ldots$, $a_m(x)$ are polynomials in x; let n be the maximum degree of any of these polynomials.

If x_1 is any number such that $a_0(x_1) \neq 0$ then the equation

$$f(x_1, y) \equiv y^m a_0(x_1) + m y^{m-1} a_1(x_1) + \ldots + a_m(x_1) = 0 \quad \ldots\ldots\ldots(1)$$

determines, by hypothesis, a single value y_1 of y. But the equation $f(x_1, y) = 0$ is of degree m in y and has therefore m roots. Accordingly these roots must each be equal to y_1, so that

$$f(x_1, y) \equiv a_0(x_1) \cdot (y - y_1)^m. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing the coefficients of y^{m-1} in (1) and (2) we see that

$$y_1 = -\frac{a_1(x_1)}{a_0(x_1)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Consider now the expression

$$\phi(x, y) \equiv [a_0(x)]^{m-1} f(x, y) - [y a_0(x) + a_1(x)]^m. \quad \ldots\ldots\ldots\ldots(4)$$

From (2) and (3) we see that $\phi(x_1, y)$ vanishes identically in y, where x_1 is an arbitrary value of x, subject only to the condition that $a_0(x_1) \neq 0$. But when $a_0(x_1) = 0$ the equation $f(x_1, y) = 0$ is of degree less than m, and has an infinite root. Hence all the roots of $f(x_1, y) = 0$ are infinite (since they are all equal) and so

$$a_1(x_1) = a_2(x_1) = \ldots = a_{m-1}(x_1) = 0.$$

Thus, from (4), $\phi(x_1, y) = 0$ in this case also.

Hence $\phi(x, y)$ vanishes identically.

Let $h(x)$ denote the highest common factor of $a_0(x)$ and $a_1(x)$, and set

$$a_0(x) = h(x) b_0(x), \quad a_1(x) = h(x) b_1(x).$$

Then from (4), on dividing by $[h(x)]^{m-1}$, we get

$$[b_0(x)]^{m-1} f(x, y) - h(x)[y b_0(x) + b_1(x)]^m \equiv 0. \quad \ldots\ldots\ldots\ldots(5)$$

Now the polynomials $a_0(x), \ldots, a_m(x)$ cannot have a common factor $k(x)$ involving x, since otherwise if x_2 is a root of $k(x_2) = 0$ the equation $f(x_2, y) = 0$ is satisfied by every value of y, whereas by hypothesis x_2 corresponds to a single value y_2 of y. Hence $h(x)$ is prime to $f(x, y)$ (unless $h(x) = 1$) and is therefore a factor of $[b_0(x)]^{m-1}$. So (5) becomes, dividing through by $h(x)$,

$$b(x) f(x, y) - [y b_0(x) + b_1(x)]^m = 0,$$

where $b(x)$ is a polynomial in x. Also, since $b(x)$ divides

$$[y b_0(x) + b_1(x)]^m$$

it must divide the coefficient of each power of y in the expansion of this. It therefore divides the highest common factor of $[b_0(x)]^m$ and $[b_1(x)]^m$. But $b_0(x)$ and $b_1(x)$ have no common factor, since $h(x)$ is the highest common factor of $a_0(x)$ and $a_1(x)$. Hence $[b_0(x)]^m$ and $[b_1(x)]^m$ have no common factor, since every linear component of such a factor must divide $b_0(x)$ and $b_1(x)$. Thus $b(x)$ must be a constant, and every pair of values (x_1, y_1) such that $f(x_1, y_1) = 0$ is such that $[y_1 b_0(x_1) + b_1(x_1)]^m$, and therefore $y_1 b_0(x_1) + b_1(x_1)$ vanishes, and vice versa. Thus the equation $f(x, y) = 0$ may be replaced by the equation

$$y b_0(x) + b_1(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which is linear in y, and of degree $n_1 \leqslant n$ in x.

We can now arrange this equation in powers of x, in the form

$$x^{n_1} c_0(y) + n_1 x^{n_1 - 1} c_1(y) + \ldots + c_{n_1}(y) = 0, \quad \ldots\ldots\ldots\ldots\ldots(7)$$

where $c_0(y), \ldots, c_{n_1}(y)$ are at most linear in y. By applying the

same argument to (7) as we did to the original equation $f(x, y) = 0$ we see that (7) can be replaced by an equation

$$xd_0(y) + d_1(y) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $d_0(y)$, $d_1(y)$ are polynomials in y of degree not exceeding the maximum degree of the polynomial $c_0(y), \ldots, c_{n_1}(y)$. Thus $d_0(y)$ and $d_1(y)$ are linear in y (or constants) and (8) is therefore the bilinear equation sought.

The above argument, though strictly elementary, may seem long. I suggest that its length shows that the tacit assumption of the theorem made in most textbooks is hardly justified.

I conclude by indicating briefly the necessary generalisation of the argument to the case in which $f(x, y) = 0$ is a polynomial equation representing an (α, β) correspondence. In this case to each value x_1 of x correspond in general β distinct values $y_1^{(1)}, \ldots, y_1^{(\beta)}$ of y (some of which may be equal for special values of x_1) while to each value y_1 of y correspond α (usually distinct) values $x_1^{(1)}, \ldots, x_1^{(\alpha)}$ of x. We may suppose that the given correspondence is irreducible. We wish to prove that the equation $f(x, y) = 0$ may be replaced by an equation $\phi(x, y) = 0$ of degree β in y and α in x.

We choose a general value x_1 of x (*i.e.* one for which the β corresponding values of y are all distinct). Then

$$f(x_1, y) \equiv a_0(x_1)(y - y_1^{(1)})^{r_1}(y - y_1^{(2)})^{r_2} \ldots (y - y_1^{(\rho)})^{r_\beta},$$

where $\Sigma r_\sigma = m$, the degree of $f(x_1, y)$ in y.

Now, by a well-known theorem of analysis, the roots of an algebraic equation are continuous functions of the coefficients. Thus, given any positive number ϵ, we can find a number δ such that if

$$|x_2 - x_1| < \delta,$$

then of the m roots of $f(x_2, y) = 0$, r_1 are at a distance $< \epsilon$ from $y_1^{(1)}$, r_2 at a distance $< \epsilon$ from $y_1^{(2)}$ and so on. If we choose ϵ to be less than half the least distance between any of the points $y_1^{(1)}, \ldots, y_1^{(\beta)}$ and remember that $f(x_1, y) = 0$ has at most β distinct roots, we see that the r_σ roots which lie near $y_1^{(\sigma)}$ must all be equal. In other words, as x_2 varies in the circle $|x_2 - x_1| < \delta$ the numbers $r_1, r_2, \ldots, r_\beta$ remain fixed.

The next step in the proof consists in showing that if the correspondence represented by the equation $f(x, y) = 0$ is irreducible, then $r_1 = r_2 = \ldots = r_\beta$. For this we suppose that the indices r_σ are arranged in descending order of magnitude, and that r_λ is the last term in the sequence which is greater than its successor, so that $r_\lambda > r_{\lambda+1} = r_{\lambda+2} = \ldots = r_\beta$. Then, for any x_2 in the circle $|x_2 - x_1| < \delta$, the expression

$$(y - y_2^{(1)})^{r_1 - r_\beta} \ldots (y - y_2^{(\lambda)})^{r_\lambda - r_\beta}$$

is the highest common factor of $f(x_2, y)$ and its first r_β derivatives with respect to y. It is therefore determinable rationally in terms

of the coefficients of $f(x_2, y)$, and its coefficients are accordingly rational functions of x_2. Thus

$$(y - y_2^{(1)})^{r_1 - r_\beta} \dots (y - y_2^{(\lambda)})^{r_\lambda - r_\beta} \equiv g(x_2, y),$$

where $g(x_2, y)$ is a polynomial in y whose coefficients are rational functions of x_2, and hence

$$f(x_2, y) \equiv g(x_2, y) \,.\, h(x_2, y),$$

where $h(x_2, y)$ is a polynomial in x_2 and y. But since this holds for every x_2 in $|x_2 - x_1| < \delta$ we must have identically

$$f(x, y) \equiv g(x, y) \,.\, h(x, y)$$

and the correspondence defined by $f(x, y)$ is reducible, one part consisting of the correspondence defined by $g(x, y) = 0$ which has a second index $\lambda < \beta$.

We can therefore suppose that $r_1 = r_2 = \dots = r_\beta = r$, whence $\beta r = m$. Hence $[a_0(x_2)]^{r-1} f(x_2, y)$ is the rth power of a polynomial in y for every x_2 in $|x_2 - x_1| < \delta$. This polynomial is determined (apart from a factor involving x_2) as the highest common factor of $f(x_2, y)$ and its first $r - 1$ derivatives with respect to y; it therefore involves x_2 rationally. Thus

$$[a_0(x_2)]^{r-1} f(x_2, y) \equiv [f_1(x_2, y)]^r$$

for all x_2 in $|x_2 - x_1| < \delta$, where $f_1(x_2, y)$ is a polynomial of degree β in y whose coefficients are rational functions (and therefore polynomials) in x_2. We must therefore have identically

$$[a_0(x)]^{r-1} f(x, y) \equiv [f_1(x, y)]^r.$$

The rest of the work proceeds as before. Removing if necessary a factor on the right involving x only we can replace $f(x, y) = 0$ by an equation of degree β in y, and an investigation shows that the degree of this in x cannot exceed the degree of $f(x, y)$ in x. Rearranging according to powers of x and proceeding as before, we are finally reduced to an equation $\phi(x, y) = 0$ of degree α in x and of degree β in y. J. A. T.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M. A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. The names of those sending the questions will not be published.

The Secretary would be glad to receive any solutions that have not yet been returned.

THE SOLUTION OF ONE-DIMENSIONAL BOUNDARY VALUE PROBLEMS BY THE LAPLACE TRANSFORMATION.

By J. C. Jaeger.

1. It is well known * that the solution of the ordinary linear differential equation with constant coefficients

$$(D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n)y = F(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where y, Dy, ... , $D^{n-1}y$ take arbitrary values y_0, y_1, ... , y_{n-1}, when

$$x = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

can be obtained by multiplying (1) by e^{-px}, ($p>0$) and integrating with regard to x from 0 to ∞.

It does not seem to have been noticed that a similar method can be used with success in cases where the independent variable has a finite range,† say $0 \leqslant x \leqslant l$, and conditions at $x = l$ take the place of some of (2) at $x = 0$.

In this Note some problems in bending of beams are treated in this way.

2. The following known results in the Integral Calculus are required :

$$\int_0^\infty e^{-px} D^n y \, dx = -y^{(n-1)}(0) - py^{(n-2)}(0) - \ldots - p^{n-1}y(0) + p^n \int_0^\infty e^{-px} y \, dx, \quad \ldots\ldots\ldots (3)$$

$$\frac{1}{n!}\int_0^\infty e^{-px} x^n \, dx = \frac{1}{p^{n+1}}, \ n \text{ a positive integer or zero.} \quad \ldots\ldots\ldots (4)$$

If

$$\left.\begin{aligned} f(x) &= 0, \quad 0<x<a, \\ &= g(x-a), \quad x>a, \end{aligned}\right\}$$

$$\int_0^\infty e^{-px} f(x)\,dx = e^{-pa}\int_0^\infty e^{-px} g(x)\,dx. \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$\int_0^\infty e^{-px} f(x)\,dx \,.\, \int_0^\infty e^{-px} g(x)\,dx = \int_0^\infty e^{-px}\,dx \int_0^x f(u)\,g(x-u)\,du. \quad \ldots. (6)‡$$

3. *Uniform beam of length l freely hinged at $x = 0$ and $x = l$; uniform load w per unit length.*

We have to solve $\quad D^4 y = \dfrac{w}{EI}, \quad 0<x<l,$

with boundary conditions $y(0) = y''(0) = y(l) = y''(l) = 0$.

* Cf. Doetsch, *Theorie und Anwendung der Laplace-Transformation* (1937). Kap. 18. Van der Pol, *Phil. Mag.* (7), 7, (1929), 1153.

The approach used here is that of Carslaw, *Math. Gazette*, 22 (1938). For the verification that solutions obtained in this way satisfy the differential equation, and conditions at $x = 0$, see Doetsch, *loc. cit.*

† But see § 7.

‡ Cf. Doetsch, *loc. cit.*, page 161.

Consider the equation $D^4y = \frac{w}{EI}$, $x>0$, ...(7)

with y, Dy, D^2y, D^3y, arbitrary, y_0, y_1, y_2, y_3, when $x=0$.

Then multiplying (7) by e^{-px}, $p>0$, and integrating with regard to x from 0 to ∞, we have, using (3) and (4), the " subsidiary " equation

$$p^4\int_0^\infty e^{-px}y\,dx = y_3 + py_2 + p^2y_1 + p^3y_0 + \frac{w}{EIp}.$$

Here we take $y_0 = y_2 = 0$ and subsequently determine y_1 and y_3 so as to make $y(l) = y''(l) = 0$. We have now

$$\int_0^\infty e^{-px}y\,dx = \frac{y_1}{p^2} + \frac{y_3}{p^4} + \frac{w}{EIp^5},$$

and so, using (4),* $\quad y = xy_1 + \frac{x^3}{3!}y_3 + \frac{w}{EI}\frac{x^4}{4!}$.

The conditions $y(l) = y''(l) = 0$ require

$$\left.\begin{aligned} \frac{l^3}{3!}y_3 + ly_1 + \frac{w}{EI}\frac{l^4}{4!} &= 0, \\ ly_3 + \frac{w}{EI}\frac{l^2}{2} &= 0. \end{aligned}\right\}$$

Whence $y_3 = -\frac{wl}{2EI}$, $y_1 = \frac{wl^3}{24EI}$, and finally

$$EI\,y = \frac{1}{24}wx^4 - \frac{1}{12}wlx^3 + \frac{1}{24}wl^3x.$$

4. *Uniform beam of length l " built in " at both ends. Loading w (constant) per unit length in $0<x<l/2$, and zero in $l/2<x<l$.*

We have to solve $D^4y = f(x)$, with $f(x) = f_1(x) + f_2(x)$, say,

where $\quad f_1(x) = \frac{w}{EI}, \quad x>0,$

and $\quad \left.\begin{aligned} f_2(x) &= 0, \quad 0<x<l/2, \\ &= -\frac{w}{EI}, \quad x>l/2. \end{aligned}\right\}$

Using (4) and (5),

$$\int_0^\infty e^{-px}f(x)\,dx = \int_0^\infty e^{-px}f_1(x)\,dx + \int_0^\infty e^{-px}f_2(x)\,dx = \frac{w}{EIp} - \frac{w}{EIp}e^{-pl/2}.$$

The boundary conditions are $y(0) = y'(0) = y(l) = y'(l) = 0$. So the

* It is known that the equation $f(p) = \int_0^\infty e^{-px}F(x)\,dx$ can have only one continuous solution $F(x)$. Cf. Doetsch, *loc. cit.*, page 35.

subsidiary equation obtained as in § 3 is

$$\int_0^\infty e^{-px} y\,dx = \frac{y_3}{p^4} + \frac{y_2}{p^3} + \frac{w}{EIp^5} - \frac{w}{EIp^5} e^{-pl/2},$$

and, by (4) and (5),

$$y = \frac{x^3}{3!}y_3 + \frac{x^2}{2!}y_2 + \frac{w}{EI}\frac{x^4}{4!} - \frac{w}{EI}\begin{cases} 0, & 0<x<l/2, \\ \dfrac{1}{4!}(x-\tfrac{1}{2}l)^4, & x>l/2. \end{cases}$$

The conditions $y(l) = y'(l) = 0$ require

$$\left.\begin{aligned} \frac{l^3}{3!}y_3 + \frac{l^2}{2!}y_2 + \frac{15}{16}\frac{w}{EI}\frac{l^4}{4!} &= 0, \\ \frac{l^2}{2!}y_3 + ly_2 + \frac{7}{8}\frac{w}{EI}\frac{l^3}{3!} &= 0. \end{aligned}\right\}$$

Hence $y_3 = -\dfrac{39}{96}\dfrac{wl}{EI}$, $y_2 = \dfrac{11}{192}\dfrac{wl^2}{EI}$, and the solution is

$$\left.\begin{aligned} EIy &= -\frac{13}{192}wlx^3 + \frac{11}{384}wl^2x^2 + \frac{1}{24}wx^4, \quad 0<x<l/2, \\ &= -\frac{13}{192}wlx^3 + \frac{11}{384}wl^2x^2 + \frac{1}{24}wx^4 - \frac{w}{24}(x-\tfrac{1}{2}l)^4, \quad x>l/2. \end{aligned}\right\}$$

5. *Uniform beam of length l with load wx per unit length in $0<x<l/2$, and $w(l-x)$ in $l/2<x<l$. The beam built in at $x=0$, and freely hinged at $x=l$.*

We have to solve $D^4y = f(x) = f_1(x) + f_2(x)$,

where $$f_1(x) = \frac{wx}{EI}, \quad x>0,$$

and

$$\left.\begin{aligned} f_2(x) &= 0, \quad 0<x<l/2, \\ &= -\frac{2w}{EI}(x-\tfrac{1}{2}l), \quad x>l/2. \end{aligned}\right\}$$

By (4) and (5), $\displaystyle\int_0^\infty e^{-px} f(x)\,dx = \frac{w}{EIp^2} - \frac{2w}{EIp^2}e^{-pl/2}$.

The boundary conditions are $y(0) = y'(0) = y(l) = y''(l) = 0$; so, proceeding as in § 3, the subsidiary equation is

$$\int_0^\infty e^{-px} y\,dx = \frac{y_3}{p^4} + \frac{y_2}{p^3} + \frac{w}{EIp^6} - \frac{2w}{EIp^6}e^{-pl/2}.$$

Thus $$y = \frac{x^3}{3!}y_3 + \frac{x^2}{2!}y_2 + \frac{w}{EI}\frac{x^5}{5!} - \frac{2w}{EI}\begin{cases} 0, & 0<x<l/2, \\ \dfrac{1}{5!}(x-\tfrac{1}{2}l)^5, & x>l/2. \end{cases}$$

The conditions at $x=l$ are

$$\left.\begin{aligned} \frac{l^3}{3!}y_3 + \frac{l^2}{2!}y_2 + \frac{15}{16}\cdot\frac{wl^5}{EI5!} &= 0, \\ ly_3 + y_2 + \frac{3}{4}\frac{wl^3}{3!EI} &= 0. \end{aligned}\right\}$$

Therefore $y_3 = -\frac{21}{128}\frac{wl^2}{EI}$, $y_2 = \frac{5}{128}\frac{wl^3}{EI}$, and finally

$$\left.\begin{aligned} EIy &= -\frac{7}{256}wx^3l^2 + \frac{5}{256}wx^2l^3 + \frac{1}{120}wx^5, \quad 0<x<l/2, \\ &= -\frac{7}{256}wx^3l^2 + \frac{5}{256}wx^2l^3 + \frac{1}{120}wx^5 - \frac{w}{60}(x-\tfrac{1}{2}l)^5, \quad l/2<x<l. \end{aligned}\right\}$$

6. *Uniform beam of length l freely hinged at its ends. The load any function of x.*

We have to solve $D^4y = \frac{f(x)}{EI}$ with boundary conditions

$$y(0) = y''(0) = y(l) = y''(l) = 0.$$

The subsidiary equation is

$$\int_0^\infty e^{-px} y\,dx = \frac{y_1}{p^2} + \frac{y_3}{p^4} + \frac{1}{EIp^4}\int_0^\infty e^{-px} f(x)\,dx,$$

Using (4) and (6),

$$y = xy_1 + \frac{x^3}{3!}y_3 + \frac{1}{3!\,EI}\int_0^x f(u)\,(x-u)^3\,du. \quad \text{.....(7)}$$

The conditions $y(l) = y''(l) = 0$ require

$$\left.\begin{aligned} ly_1 + \frac{1}{6}l^3y_3 + \frac{1}{6EI}\int_0^l f(u)\,(l-u)^3\,du = 0, \\ ly_3 + \frac{1}{EI}\int_0^l f(u)\,(l-u)\,du = 0. \end{aligned}\right\} \quad \text{.....(8)}$$

Thus $$ly_1 = \frac{1}{6EI}\int_0^l f(u)\,(l-u)\,(2lu-u^2)\,du,$$

and substituting in (7), we have

$$y = \frac{(l-x)}{6EIl}\int_0^x u\,(2lx - x^2 - u^2)f(u)\,du + \frac{x}{6EIl}\int_x^l (l-u)\,(2lu - x^2 - u^2)f(u)\,du, \quad 0<x<l.$$

In particular if the load is W, concentrated * at x', so that $f(x)$ vanishes except in a small region about x', we have

$$\left.\begin{aligned} y &= \frac{W}{6EIl}x'\,(l-x)\,(2lx - x^2 - x'^2), \quad l>x>x' \\ &= \frac{W}{6EIl}x\,(l-x')\,(2lx' - x^2 - x'^2), \quad 0<x<x' \end{aligned}\right\} \quad \text{.....(9)}$$

* Concentrated loads are most easily discussed by the use of the Dirac δ function, $\delta(x-x')$, defined as zero if $x \neq x'$, and infinite at $x = x'$ in such a way that

$$\int_{-\infty}^{\infty} \delta(x-x')\,dx = 1.$$

This has the property

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx = f(x') \quad \text{and so} \quad \int_0^\infty e^{-px}\,\delta(x-x')\,dx = e^{-px'}.$$

7. In all the problems considered we have taken $f(x)$ as defined for $x>l$ by the formula which defines it for $0<x<l$. Now in the problem of § 6 the equations (7) and (8) which determine y in $0<x<l$ do not involve the values of $f(x)$ for $x>l$; so we may take any value we please for $f(x)$ in $x>l$ without affecting the result in $0<x<l$. This remark will apply to any boundary conditions at 0 and l. But, by changing the value of $f(x)$ in $x>l$, we may change the form of the solution in $0<x<l$.

For example, if in the problem of § 3, in place of choosing $f(x)=w$ for $x>l$, we give it a periodic value, we obtain a Fourier series for y.

Let $\left.\begin{aligned} f(x)&=w, \quad 2rl<x<(2r+1)l \\ &=-w, \quad (2r+1)l<x<(2r+2)l \end{aligned}\right\}$ $r=0, 1, 2 \ldots$.

Then using (5),

$$\int_0^\infty e^{-px} f(x)\,dx = \frac{w}{p}\{1-2e^{-pl}+2e^{-2pl}-\ldots\}$$

$$= \frac{w}{p}\tanh\frac{pl}{2}.$$

We have to solve $D^4y = \dfrac{f(x)}{EI}$, with boundary conditions

$$y(0)=y''(0)=y(l)=y''(l)=0.$$

The subsidiary equation is

$$\int_0^\infty e^{-px} y\,dx = \frac{y_3}{p^4}+\frac{y_1}{p^2}+\frac{w}{EIp^5}\tanh\frac{pl}{2},$$

whence using the inversion formula * for the last term

$$y = \frac{x^3}{3!}y_3 + xy_1 + \frac{w}{2\pi iEI}\int_{c-i\infty}^{c+i\infty} e^{\lambda x}\tanh\frac{\lambda l}{2}\,\frac{d\lambda}{\lambda^5}$$

$$= \frac{x^3}{3!}y_3 + xy_1 + \frac{w}{EI}\left(\frac{lx^3}{12}-\frac{xl^3}{24}\right) + \frac{4wl^4}{\pi^5 EI}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\sin(2n+1)x\pi/l.$$

Then for a concentrated load W at x' we have to solve $D^4y = W\delta(x-x')/(EI)$, which has the subsidiary equation

$$p^4\int_0^\infty e^{-px} y\,dx = y_3+py_2+p^2y_1+p^3y_0+(W/EI)e^{-px'},$$

and introducing the boundary conditions of this section and proceeding as before we easily obtain (9).

* That (if $f(p)$ satisfies certain conditions) when

$$f(p)=\int_0^\infty e^{-pt}F(t)\,dt, \quad F(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}f(\lambda)\,d\lambda, \quad c>0.$$

The line integral is evaluated in this case by completing the contour by a semi-circle in the left-hand half plane not passing through any pole of the integrand; it is easy to show that the integral over the semi-circle vanishes as its radius $\to\infty$, and evaluating the residues at the poles of the integrand

$$(\lambda=0 \text{ and } \pm(2n+1)\pi i/l, \quad n=0, 1, \ldots)$$

gives the result stated.

See Carslaw, *loc. cit.*, for a detailed discussion of the method.

At $x=l$ we require $\left.\begin{aligned}\frac{l^3}{3!}y_3+ly_1+\frac{wl^4}{24EI}&=0,\\ ly_3+\frac{wl^2}{2EI}&=0.\end{aligned}\right\}$

So $y_3=-\dfrac{wl}{2EI}$ and $y_1=\dfrac{wl^3}{24EI}$, so

$$y=\frac{4wl^4}{\pi^5EI}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\sin(2n+1)\pi x/l,$$

which is the Fourier series for the deflection.* J. C. J.

INTERCHANGE OF TEACHERS BETWEEN GREAT BRITAIN AND THE UNITED STATES.

THE interchange of salaried appointments between masters and mistresses in British and American schools is arranged in Great Britain by a Committee representative of the English-Speaking Union, the British Federation of University Women, the Incorporated Association of Headmistresses, the Headmasters' Conference, and the Association of Headmasters, working in conjunction with Committees in the United States.

The appointments are open only to teachers holding suitable qualifications, who at the time of their application are holding a position in a school in Great Britain and are arranged on the understanding that they will return to their original posts on the completion of their year's work in the United States. Any subjects in the usual school curriculum may be offered, including physical education and nursery school work. The arrangements for interchange have been approved by the Board of Education, and service overseas is regarded as contributory service for superannuation purposes and it may also count for salary increments under the Burnham Scale.

Such interchange appointments enable British teachers to gain practical experience in another country and, as the summer term in the United States ends in June, the long vacation offers interesting possibilities for travel and research.

All applications for interchange appointments for the year 1939-1940 must reach the Committee not later than March 31st, 1939.

Further information regarding the type of school, salaries, income tax, etc., may be obtained from the Honorary Secretary, British Joint Committee for Interchange, c/o The English-Speaking Union, Dartmouth House, 37 Charles Street, Berkeley Square, London, W. 1.

* Timoshenko and Lascelles, *Applied Elasticity*, p. 230.

1246. But now still remains that terrible algebra and mathematics; after Christmas it is necessary that I have lessons in those also. . . . If one says or writes a word amiss at an examination, that may be the cause of failure. May God give me the wisdom which I need and grant me what I so fervently desire: to finish my studies as quickly as possible and be ordained, so that I can perform the practical duties of a clergyman.—Vincent Van Gogh to his brother, May, 1877. [Per Mr. E. H. Lockwood.]

A RELATED TSCHEBYSCHEFF POLYNOMIAL.

By J. P. M'Carthy.

1. We know that

$$\log(1-2x\cos\theta+x^2) = -2\left(x\cos\theta+\frac{x^2}{2}\cos 2\theta+\ldots+\frac{x^n}{n}\cos n\theta+\ldots\right), \quad \ldots(1)$$

the series being absolutely and uniformly convergent for all values of θ and for $|x|<1$.

Now write μ for $\cos\theta$ on the left and the coefficients of the powers of x on the right are all functions of μ.

Hence, $$\log(1-2\mu x+x^2)=\sum_{n=1}^{n=\infty} x^n Q_n(\mu), \quad \ldots\ldots(2)$$

where $$Q_n(\mu)=-\frac{2}{n}\cos n\theta \text{ and } -1\leqslant\mu\leqslant 1. \quad \ldots\ldots(3)$$

Also, $$\log(a^2-2\mu ar+r^2)=2\log a+\log\left[1-2\mu\left(\frac{r}{a}\right)+\left(\frac{r}{a}\right)^2\right]$$

$$=2\log a+\sum_{n=1}^{n=\infty} Q_n(\mu)\left(\frac{r}{a}\right)^n, \quad \ldots\ldots(4)$$

where $$|r/a|<1.$$

If $|a/r|<1$ there is a similar expansion.

In (2) put $\mu=1$ and $2\log(1-x)=\sum_{n=1}^{n=\infty} x^n Q_n(1)$.

But $$2\log(1-x)=-2\sum_{n=1}^{n=\infty}\frac{x^n}{n}.$$

Thus $$Q_n(1)=-2/n,\ n>0. \quad \ldots\ldots(5)$$

Similarly $$Q_n(-1)=(-1)^{n-1}2/n,\ n>0, \quad \ldots\ldots(6)$$

whence $$Q_n(-1)=(-1)^n Q_n(1). \quad \ldots\ldots(7)$$

In general, $$1-2(-\mu)x+x^2=1-2\mu(-x)+x^2.$$

Hence $$\Sigma x^n Q_n(-\mu)=\Sigma(-x)^n Q_n(\mu)$$

and $$Q_n(-\mu)=(-1)^n Q_n(\mu). \quad \ldots\ldots(8)$$

If in (2) we put $\mu=0$ and expand both sides, we deduce that

$$Q_{2n+1}(0)=0 \text{ and } Q_{2n}(0)=(-1)^{n-1}/n. \quad \ldots\ldots(9)$$

2. The Recurrence Formula.

Let $$V=\log(a^2-2\mu ar+r^2)$$

$$=2\log a+Q_1(\mu)\cdot\frac{r}{a}+Q_2(\mu)\left(\frac{r}{a}\right)^2+\ldots+Q_n(\mu)\left(\frac{r}{a}\right)^n+\ldots.$$

Then $$\frac{\partial V}{\partial a}=\frac{2a-2\mu r}{a^2-2\mu ar+r^2}, \quad \frac{\partial V}{\partial r}=\frac{2r-2\mu a}{a^2-2\mu ar+r^2},$$

whence $$(r-\mu a)\frac{\partial V}{\partial a}=(a-\mu r)\frac{\partial V}{\partial r},$$

i.e. $$(r-\mu a)\left[\frac{2}{a}-Q_1(\mu)\frac{r}{a^2}-Q_2(\mu)\frac{2r^2}{a^3}-\ldots-Q_n(\mu)\frac{nr^n}{a^{n+1}}-\ldots\right]$$
$$=(a-\mu r)\left[Q_1(\mu)\frac{1}{a}+Q_2(\mu)\frac{2r}{a^2}+\ldots+Q_n(\mu)\frac{nr^{n-1}}{a^n}+\ldots\right].$$

Equating coefficients of r^n, we have

$$(n+1)Q_{n+1}(\mu)-2\mu nQ_n(\mu)+(n-1)Q_{n-1}(\mu)=0.$$

The same result is obtained if we expand V in terms of a/r and equate, finally, coefficients of a^n.

3. Calculation of the Functions $Q_n(\mu)$.

In the previous section put $a=1$ for convenience, and equate the two series we have for V.

Then $$-(2\mu r-r^2)-\tfrac{1}{2}(2\mu r-r^2)^2-\tfrac{1}{3}(2\mu r-r^2)^3-\ldots$$
$$=rQ_1(\mu)+r^2Q_2(\mu)+r^3Q_3(\mu)+\ldots$$

whence $$Q_1(\mu)=2\mu \text{ and } Q_2(\mu)=1-2\mu^2. \quad \ldots\ldots\ldots\ldots\ldots(a)$$

Now, putting $x=2$ in the recurrence formula, we find

$$Q_3(\mu)=-\tfrac{1}{3}(8\mu^3-6\mu)=-\tfrac{2}{3}(4\mu^3-3\mu). \quad \ldots\ldots\ldots\ldots\ldots(b)$$

Similarly, continuing the process, we find in succession:

$$Q_4(\mu)=-\tfrac{1}{2}(8\mu^4-8\mu^2+1)=-\tfrac{2}{4}(8\mu^4-8\mu^2+1), \quad \ldots\ldots\ldots\ldots\ldots(c)$$
$$Q_5(\mu)=-\tfrac{2}{5}(16\mu^5-20\mu^3+5\mu), \quad \ldots\ldots\ldots\ldots\ldots(d)$$
$$Q_6(\mu)=-\tfrac{1}{3}(32\mu^6-48\mu^4+18\mu^2-1)=-\tfrac{2}{6}(32\mu^6-48\mu^4+18\mu^2-1), \quad (e)$$
$$Q_7(\mu)=-\tfrac{2}{7}(64\mu^7-112\mu^5+56\mu^3-7\mu), \quad \ldots\ldots\ldots\ldots\ldots(f)$$

and so on. The functions of μ give, in each case, the expansion of the cosine of a multiple of θ in powers of $\cos\theta$.

4. In the previous section equate, from the two series for V, the coefficients of r^n and

$$Q_n(\mu)=-\frac{(2\mu)^n}{n}+\frac{(n-1)(2\mu)^{n-2}}{n-1}-\frac{(n-2)(n-3)(2\mu)^{n-4}}{(n-2)\,.\,2!}$$
$$+\frac{(n-3)(n-4)(n-5)(2\mu)^{n-6}}{(n-3)\,.\,3!}$$
$$-\frac{(n-4)(n-5)(n-6)(n-7)(2\mu)^{n-8}}{(n-4)\,.\,4!}+\ldots$$
$$+(-1)^{r+1}\frac{(n-r)(n-r-1)(n-r-2)\ldots(n-2r+1)(2\mu)^{n-2r}}{(n-r)\,.\,r!}+\ldots$$

$$= -\frac{1}{n}\Big[(2\mu)^n - \frac{n(2\mu)^{n-2}}{1!} + \frac{n(n-3)(2\mu)^{n-4}}{2!} - \frac{n(n-4)(n-5)}{3!}(2\mu)^{n-6}$$

$$+ \ldots + (-1)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}(2\mu)^{n-2r} + \ldots\Big].$$

But $Q_n(\mu) = -\frac{2}{n}\cos n\theta$, from which it follows that

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \ldots$$

$$+ (-1)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}(2\cos\theta)^{n-2r} + \ldots.$$

5. As an alternative method we might have started in sec. 1 by taking equation (2) as the fundamental assumption without assuming equations (1) or (3). The results (5) to (9) of that section follow. In sec. 4 we should find the series for $Q_n(\mu)$, and assuming at that point that the *series* in the square bracket above is equal to $2\cos n\theta$, we have $Q_n(\mu) = -\frac{2}{n}\cos n\theta$.

6. Differential Coefficients.

From $V = \log(1 - 2\mu r + r^2)$, we have

$$(\mu - r)\frac{\partial V}{\partial \mu} = r\frac{\partial V}{\partial r}.$$

But $V = Q_1(\mu)r + Q_2(\mu)r^2 + \ldots + Q_n(\mu)r^n + \ldots$.

Hence, dashes denoting differentiation with respect to μ,

$$(\mu - r)[rQ_1'(\mu) + r^2Q_2'(\mu) + \ldots + r^{n-1}Q_{n-1}'(\mu) + r^nQ_n'(\mu) + \ldots]$$
$$= r[Q_1(\mu) + 2rQ_2(\mu) + \ldots + nr^{n-1}Q_n(\mu) + \ldots].$$

Equating coefficients of r^n, we have

$$nQ_n(\mu) = \mu Q_n'(\mu) - Q_{n-1}'(\mu).$$

This relation is similar to that satisfied by the Legendre coefficient.

7. Other Recurrence Formulae.

We prove that $Q_{n+1}'(\mu) - Q_{n-1}'(\mu) = 2nQ_n(\mu)$, ...(1)

$$Q_{n+1}'(\mu) - \mu Q_n'(\mu) = nQ_n(\mu), \quad \ldots\ldots(2)$$

$$(1-\mu^2)Q_n'(\mu) = \mu nQ_n(\mu) - (n+1)Q_{n+1}(\mu). \quad (3)$$

We have seen that

$$(n+1)Q_{n+1}(\mu) - 2\mu nQ_n(\mu) + (n-1)Q_{n-1}(\mu) = 0 \quad \ldots\ldots(a)$$

and

$$nQ_n(\mu) = \mu Q_n'(\mu) - Q_{n-1}'(\mu). \quad \ldots\ldots(b)$$

Differentiate (a) and substitute for $\mu Q_n'(\mu)$ from (b): the result (1) follows.

Subtract (*b*) from (1) and

$$Q'_{n+1}(\mu) - \mu Q_n'(\mu) = nQ_n(\mu). \quad \text{.....(2)}$$

In (2), put $(n-1)$ for n and

$$Q_n'(\mu) - \mu Q'_{n-1}(\mu) = (n-1)\,Q_{n-1}(\mu).$$

From (*b*), $\quad \mu^2 Q_n'(\mu) - \mu Q'_{n-1}(\mu) = \mu n Q_\mu(\mu).$

Thus $(1-\mu^2)\,Q_n'(\mu) = (n-1)\,Q_{n-1}(\mu) - \mu n Q_n(\mu)$

$$= 2\mu n Q_n(\mu) - (n+1)\,Q_{n+1}(\mu) - \mu n Q_n(\mu) \text{ from } (a)$$

$$= \mu n Q_n(\mu) - (n+1)\,Q_{n+1}(\mu). \quad \text{.....(3)}$$

8. The differential equation satisfied by $Q_n(\mu)$.

Any constant multiple of $\cos n\theta$ satisfies the equation

$$\frac{d^2y}{d\theta^2} + n^2 y = 0.$$

Put $\mu = \cos\theta$, and

$$\frac{dy}{d\theta} = \frac{dy}{d\mu}\cdot\frac{d\mu}{d\theta} = -\sin\theta\,\frac{dy}{d\mu},$$

$$\frac{d^2y}{d\theta^2} = -\cos\theta\,\frac{dy}{d\mu} + \sin^2\theta\,\frac{d^2y}{d\mu^2}.$$

The differential equation now becomes

$$\sin^2\theta\,\frac{d^2y}{d\mu^2} - \cos\theta\,\frac{dy}{d\mu} + n^2 y = 0\,;$$

or,
$$(1-\mu^2)\,\frac{d^2y}{d\mu^2} - \mu\,\frac{dy}{d\mu} + n^2 y = 0. \quad \text{.....(1)}$$

Hence $(1-\mu^2)Q_n''(\mu) - \mu Q_n'(\mu) + n^2 Q_n(\mu) = 0.$ .....(2)

[The result (2) can be obtained easily by differentiating (3) of sec. 7 and using (2) of the same section.]

9. The Orthogonal Property.

If m and n are positive integers, we know that

$$\int_0^\pi \cos m\theta \cos n\theta\, d\theta = 0 \quad \text{if } m \neq n$$

$$= \frac{\pi}{2} \quad \text{if } m = n.$$

Now
$$\int_{-1}^{1} \frac{Q_n(\mu)Q_m(\mu)}{\sqrt{(1-\mu^2)}}\,d\mu = \frac{4}{mn}\int_0^\pi \cos n\theta \cos m\theta\, d\theta$$

where $\mu = \cos\theta$

$$= 0 \quad \text{if } m \neq n \quad \text{.....(1)}$$

$$= \frac{2\pi}{n^2} \quad \text{if } m = n. \quad \text{.....(2)}$$

The result (1) can also be derived from the differential equation thus: the equation may be written

$$\frac{d}{d\mu}\left(\sqrt{(1-\mu^2)}\frac{dQ_n(\mu)}{d\mu}\right)=-\frac{n^2Q_n(\mu)}{\sqrt{(1-\mu^2)}}.$$

Hence $$Q_m(\mu)\frac{d}{d\mu}\left(\sqrt{(1-\mu^2)}\frac{dQ_n(\mu)}{d\mu}\right)=-\frac{n^2Q_n(\mu)Q_m(\mu)}{\sqrt{(1-\mu^2)}},$$

and $$Q_n(\mu)\frac{d}{d\mu}\left(\sqrt{(1-\mu^2)}\frac{dQ_m(\mu)}{d\mu}\right)=-\frac{m^2Q_m(\mu)Q_n(\mu)}{\sqrt{(1-\mu^2)}}.$$

Subtracting and integrating by parts over the range from -1 to 1, we find that

$$(m^2-n^2)\int_{-1}^{1}\frac{Q_m(\mu)Q_n(\mu)}{\sqrt{(1-\mu^2)}}d\mu=0;$$

and hence that, if $m\neq n$, $\int_{-1}^{1}\frac{Q_m(\mu)Q_n(\mu)}{\sqrt{(1-\mu^2)}}d\mu=0.$

This method is analogous to a common method of proving the corresponding orthogonal property of the Legendre polynomials.

COROLLARY. $$\int_0^1[Q_n(\mu)]^2\frac{d\mu}{\sqrt{(1-\mu^2)}}=\frac{\pi}{n^2}.$$

10. To evaluate $\int_{-1}^{1}\frac{\mu^rQ_n(\mu)}{\sqrt{(1-\mu^2)}}d\mu.$

It is known that

$$\int_0^\pi\cos^n\theta d\theta=0,\quad n \text{ odd},$$

$$=\frac{(n-1)(n-3)(n-5)\ldots3.1}{n(n-2)(n-4)\ldots4.2}\pi,\quad n \text{ even}.$$

Putting $\mu=\cos\theta$, we have

$$I=\int_{-1}^{1}\frac{\mu^rQ_n(\mu)}{\sqrt{(1-\mu^2)}}d\mu=-\frac{2}{n}\int_0^\pi\cos^r\theta\cos n\theta d\theta$$

$$=-\frac{1}{n}\sum_{s=0}(-1)^s\int_0^\pi\frac{(n-s-1)!n}{s!(n-2s)!}2^{n-2s}\cos^{n+r-2s}\theta d\theta$$

$$=\sum_{s=0}(-1)^{s-1}\frac{(n-s-1)!}{s!(n-2s)!}2^{n-2s}\int_0^\pi\cos^{n+r-2s}\theta d\theta.$$

Hence, if $n+r$ is odd, $I=0$; and, if $n+r$ is even, we have

$$I=\Pi\sum_{s=0}(-1)^{s-1}2^{n-2s}\frac{(n-s-1)!}{s!(n-2s)!}$$

$$\times\frac{(n+r-2s-1)(n+r-2s-3)\ldots3.1}{(n+r-2s)(n+r-2s-2)\ldots4.2}.$$

When n is even, the summation extends from $s=0$ to $s=\frac{n}{2}$, and to $s=\frac{n-1}{2}$ when n is odd.

11. Expression for a Polynomial as a Linear Function of the Polynomials $Q_n(x)$.

The polynomial

$$f(x)\equiv A_0x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots+A_{n-1}x+A_n \quad \ldots\ldots\ldots(1)$$

can be expressed in the form

$$f(x)=B_nQ_n(x)+B_{n-1}Q_{n-1}(x)+\ldots+B_1Q_1(x). \quad \ldots\ldots\ldots\ldots(2)$$

The coefficient of x^n in $Q_n(x)$ is $-\frac{2^n}{n}$; therefore the expression $f(x)+A_0\frac{n}{2^n}Q_n(x)$ is a polynomial $f_1(x)$ of degree $(n-1)$ at most.

Hence, if $B_n=-\frac{n}{2^n}A_0$, we have

$$f(x)=B_nQ_n(x)+f_1(x).$$

Similarly, $f_1(x)=B_{n-1}Q_{n-1}(x)+f_2(x)$ where $f_2(x)$ is a polynomial of degree $(n-2)$ at most.

Hence, $$f(x)=B_nQ_n(x)+B_{n-1}Q_{n-1}(x)+f_2(x).$$

Continuing the process, we can express $f(x)$ in the required form. The coefficient B_r can be obtained thus :

Multiply through (2) by $\frac{Q_r(x)}{\sqrt{(1-x^2)}}$ and integrate from -1 to 1.

Then

$$\int_{-1}^{1}f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx=B_r\int_{-1}^{1}\frac{[Q_r(x)]^2}{\sqrt{(1-x^2)}}\,dx=\frac{2\pi}{r^2}B_r \text{ from sec. 9,}$$

and $B_r=\frac{r^2}{2\pi}\int_{-1}^{1}f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx.$

If we write $x=\cos\theta$, then

$$\int_{-1}^{1}f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx=-\frac{2}{r}\int_0^{\pi}f(\cos\theta)\cos r\theta\,d\theta$$

and

$$B_r=-\frac{r}{\pi}\int_0^{\pi}f(\cos\theta)\cos r\theta\,d\theta.$$

Since $f(\cos\theta)$ is a polynomial in $\cos\theta$, these integrals may be calculated from sec. 10.

12. Expansion of a function in a series of Polynomials $Q_n(x)$.

Let us assume that the function $f(x)$ satisfies Dirichlet's conditions in the interval $(-1, 1)$ and that it can be expanded in the series

$$f(x)=A_1Q_1(x)+A_2Q_2(x)+\ldots+A_rQ_r(x)+\ldots, \quad \ldots\ldots\ldots(1)$$

where the coefficients A_r are constants.

If we assume also that we may integrate term by term over the range $(-1, 1)$, the coefficient A_r is given by

$$\int_{-1}^{1} f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx = A_r\int_{-1}^{1}\frac{[Q_r(x)]^2}{\sqrt{(1-x^2)}}\,dx = \frac{2\pi}{r^2}A_r,$$

and
$$A_r = \frac{r^2}{2\pi}\int_{-1}^{1} f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx. \quad \ldots\ldots\ldots\ldots(2)$$

Now, write $x = \cos\theta$; then

$$f(x) = f(\cos\theta) = -2[A_1 + \tfrac{1}{2}A_2\cos 2\theta + \ldots + \frac{1}{r}A_r\cos r\theta + \ldots]$$

$$= -2\sum_{r=1}^{r=\infty}\frac{1}{r}A_r\cos r\theta. \quad \ldots\ldots\ldots\ldots(3)$$

We saw in sec. 11 that

$$\int_{-1}^{1} f(x)\frac{Q_r(x)}{\sqrt{(1-x^2)}}\,dx = -\frac{2}{r}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta,$$

whence
$$A_r = -\frac{r}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta\,; \quad \ldots\ldots\ldots\ldots(4)$$

and, from (3),

$$f(x) = f(\cos\theta) = \frac{2}{\pi}\sum_{r=1}^{r=\infty} Q_r(x)\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta. \quad \ldots\ldots\ldots\ldots(5)$$

It appears, then, that the conditions to be satisfied by the function are similar to those necessary to its expansion in a Fourier series.

13. The differential equation and its solution.

(*a*) We saw that $Q_n(x)$ satisfies the equation

$$(1-x^2)\frac{d^2z}{dx^2} - x\frac{dz}{dx} + n^2z = 0. \quad \ldots\ldots\ldots\ldots(1)$$

It is easy to show that if we differentiate the equation

$$(1-x^2)\frac{d^2z_m}{dx^2} + (2m-1)x\frac{dz_m}{dx} + (n^2-m^2)z_m = 0 \quad \ldots\ldots\ldots\ldots(2)$$

m times and write $z = \dfrac{d^m z_m}{dx^m}$, we get equation (1).

The equation (2) is of the form

$$(1-x^2)\frac{d^2y}{dx^2} + (2m-1)x\frac{dy}{dx} + (n^2-m^2)y = 0. \quad \ldots\ldots\ldots\ldots(3)$$

If $n = m$, equation (3) becomes

$$(1-x^2)\frac{d^2y}{dx^2} + (2n-1)x\frac{dy}{dx} = 0. \quad \ldots\ldots\ldots\ldots(4)$$

If we differentiate (4) n times we get an equation of the same form as (1), so that the nth derivative of any function of x which satisfies (4) is a solution of (1).

It is easily seen that (4) is satisfied by

$$\frac{dy}{dx} = C(1-x^2)^{\frac{1}{2}(2n-1)}$$

where C is a constant.

Hence (1) is satisfied by $z = C\dfrac{d^{n-1}}{dx^{n-1}}(1-x^2)^{\frac{1}{2}(2n-1)}$.

But, clearly, $Q_n(x) \neq \dfrac{d^{n-1}}{dx^{n-1}}(1-x^2)^{\frac{1}{2}(2n-1)}$.

(*b*) Now let us consider the solution in series of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0. \quad \text{.....(5)}$$

Assuming the trial solution, where $a_0 \neq 0$,

$$y = x^c(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots),$$

we find the indicial equation $a_0c(c-1) = 0$,

whence $\qquad c = 0 \quad \text{or} \quad c = 1.$

(i) If $c = 1$, we find that $a_1 = 0 = a_3 = \ldots = a_{2n+1} = \ldots$, and

$$y = ax_0\left[1 - \frac{n^2-1^2}{3!}x^2 + \frac{(n^2-1^2)(n^2-3^2)}{5!}x^4 - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{7!}x^6 + \ldots\right]. \quad \text{.....(6)}$$

(ii) If $c = 0$, a_1 is indeterminate and

$$y = a_0\left[1 - \frac{n^2x^2}{2!} + \frac{n^2(n^2-2^2)}{4!}x^4 - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + \ldots\right] + a_1\left[x - \frac{n^2-1^2}{3!}x^3 + \frac{(n^2-1^2)(n^2-3^2)}{5!}x^5 - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{7!}x^7 + \ldots\right], \quad \text{.....(7)}$$

which contains two arbitrary constants and includes the solution (6).

If n is even, the first series in (7) terminates and is equal to $(-1)^{n/2}\cos n\theta$ where $x = \cos\theta$,

that is, equal to $\qquad (-1)^{\frac{n+2}{2}}\dfrac{n}{2}Q_n(x).$

Hence in this case the solution is

$$y = A_0 Q_n(x) + B_0 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{\frac{1}{2}(2n-1)}.$$

Similarly, if n is odd, the second series in (7) terminates and the solution is

$$y = A_1 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{\frac{1}{2}(2n-1)} + B_1 Q_n(x).$$

14. Relation to the Tschebyscheff Polynomials.

It is well known that

(*a*) if we write $\dfrac{1-r^2}{1-2rx+r^2} = \sum_{n=0}^{n=\infty} T_n(x)(2r)^n$,

where $|r|<1$ and $-1 \leqslant x \leqslant 1$, the coefficients $T_n(x)$ are the Tschebyscheff Polynomials;

(*b*) $T_0(x)=1$, $T_n(x) = \dfrac{1}{2^{n-1}} \cos n\theta$ where $x = \cos\theta$;

(*c*) the recurrence formula is

$$T_{n+1}(x) - xT_n(x) + \tfrac{1}{4}T_{n-1}(x) = 0;$$

(*d*) $T_n(x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0,$$

which is the same as that satisfied by our function $Q_n(x)$;

(*e*) with the aid of the factor $1/(1-x^2)^{\frac{1}{2}}$ the functions $T_n(x)$ constitute an orthogonal system; that is,

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{(1-x^2)}} = 0, \quad n \neq m.$$

We see, then, that $2^n T_n(x) = -nQ_n(x)$, and that properties of the one type of polynomial may be inferred from those of the other.

J. P. McC.

1247. Finite Differences.

Interpolations in the Japanese Diet to-day became heated, particularly on the subject of the Government's economic policies, and the Government came to the conclusion that the questioners were "belligerent".—*Daily Mail*, January 24, 1938. [Per. Professor L. M. Milne-Thomson.]

1248. Which Part?

Did not Scotland beat Wales, who beat England, who, in their turn, lambasted Ireland by a total which sounded like part of the differential calculus?—*Daily Telegraph*, February 24, 1938. [Per Mr. H. B. Hodgson.]

1249. "It seems to me so unlikely as to be virtually impossible that any skin would show no mark of the impact of the full weight of the body, which would be multiplied by the momentum, against the edges of the stairs."—J. S. Fletcher and "Torquemada", *Todmanhawe Grange*. [Per Mr. W. J. Hodgetts.]

MATHEMATICAL NOTES.

1345. "*A problem in cartophily.*"

Dr. Maunsell's article under this title in the October *Gazette* has aroused considerable interest. This note gives the names of those who have contributed, and a summary of results.

Using the notation of the article, it can be shown that

$$f(r) = r\left(1 + \frac{1}{2} + \ldots + \frac{1}{r}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and hence the asymptotic expansion is

$$f(r) \sim r \log r + \gamma r + \tfrac{1}{2} - \sum_{n=1}^{\infty} (-)^{n-1} B_n / 2n r^{2n-1}. \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Mr. M. G. Kendall gave the formula (i) in *The New Statesman* (July 9, 1938, p. 94; July 23, 1938, p. 168), and a proof appeared in an appendix to a paper in *Biometrika*, December, 1938. Mr. F. J. Cock gives a reference to Whitworth, *DCC Exercises in Choice and Chance*, where two solutions of an analogous problem (No. 685) are given, and to De Moivre, *The Doctrine of Chances*, Problems XXXIX and XL.

Whitworth's two solutions, restated in the terms of the present problem, are:

(A) If p fresh cards are wanted to make up the set of r, it is necessary, on the average, to purchase r/p packets to get one of these cards. Thus, on the average, the number of packets to be purchased to make up the complete set is

$$r\left(\frac{1}{r} + \frac{1}{r-1} + \ldots + \frac{1}{2} + 1\right).$$

(B) If E_s is the expectation when s different cards have been collected and r the total number of cards,

$$E_s = \frac{r-s}{r}(1 + E_{s+1}) + \frac{s}{r}(1 + E_s),$$

or

$$E_s = r/(r-s) + E_{s+1},$$

and since $E_r = 0$, we have

$$E_0 = r\left(1 + \frac{1}{2} + \ldots + \frac{1}{r}\right).$$

Mr. F. Ayres, Mr. R. Walker and Dr. F. J. W. Whipple sent proofs of (i) on the lines of (A); Mr. R. E. Beard sent a proof of type (B); Mr. D. Burnett, Mr. R. C. Chevalier and Mr. H. J. Downing also supplied proofs varying in detail, but using the principle underlying (A) and (B).

Mr. R. E. Beard, Dr. R. Cooper, Mr. R. L. Goodstein, Mr. R. Walker and Dr. F. J. W. Whipple have shown that Dr. Maunsell's

formula (2) for $f(v)$ is equivalent to (i) above by showing that his formula gives

$$f(r) = r^2\left\{1 - \binom{r-1}{1}\frac{1}{2^2} + \binom{r-1}{2}\frac{1}{3^2} + \ldots \pm \frac{1}{(r-1)^2}\right\}$$

$$= r\int_0^1 \{1-(1-x)^r\}/x \,.\, dx$$

$$= r\int_0^1 (1 + u + u^2 + \ldots + u^{r-1})\,du$$

$$= r\left(1 + \frac{1}{2} + \ldots + \frac{1}{r}\right).$$

Mr. M. G. Kendall supplied a proof of this on slightly different lines. Mr. M. D. McCarthy shows that $D(n, r) = r^{-n+1}\Delta^{r-1}0^{n-1}$, and hence that $f(r) = r^2\Delta^{r-1}[z^{-2}]_{z=-r}$, whence evaluating the difference of a product, (i) is obtained: he gives the asymptotic expansion (ii).

Mr. R. L. Goodstein gave a simple algebraic proof of the equivalence of Dr. Maunsell's two formulae for $f(r)$.

Prof. G. N. Watson, in the following note, proves that Dr. Maunsell's formulae (1) and (2) for $f(r)$ are equivalent to (i) above; the proof uses the theory of residues. He also quotes the asymptotic expansion (ii) above.

Mr. Beard and Dr. Whipple also point out that "the most probable number" is the "mode" of the statistician.

Mr. G. R. Langdale refers to J. V. Uspensky, *Introduction to Mathematical Probability*, where an answer, though not the details, can be found, together with some results of experiments.

1346. *A postscript to Maunsell's problem in cartophily.*

In his interesting paper in the *Gazette* (vol. 22, pp. 328-331), Maunsell has investigated the expectation of the number of packets of cigarettes to be bought in order to obtain a complete set of r cigarette cards; and he has shown that, if this expectation is denoted by $f(r)$, then $f(r)$ is expressible in each of the following forms:

$$f(r) = \sum_{n=1}^{r-1}(-)^{n-1}\binom{r-1}{n-1}\frac{(r-n)^{r-1}}{r^{r-2}\,n^2}(nr + r - n),$$

$$f(r) = \sum_{n=1}^{r-1}(-)^{n-1}\binom{r}{n}\frac{(r-n)^r}{r^{r-1}\,n} + r.$$

With the help of the calculus of residues, it is possible to derive a simpler expression for $f(r)$ from these results. Consider the function $F(z)$ of the complex variable z defined by the formula

$$F(z) = -\frac{(r-1)!\,(r-z)^{r-1}(rz + r - z)}{r^{r-2}\,z^2(1-z)(2-z)\ldots(r-z)}.$$

The only singularities of $F(z)$ are simple poles at the points 1, 2, ...,

$r-1$, and a double pole at the origin. The residue at any pole $z=n$ (the origin excepted) is equal to

$$(-)^{n-1}\binom{r-1}{n-1}\frac{(r-n)^{r-1}}{r^{r-2}\,n^2}(nr+r-n),$$

and so, from the first of Maunsell's expressions, we see that $f(r)$ is equal to the sum of the residues of $F(z)$ at the poles $1, 2, \ldots, r-1$.

Now, for large values of $|z|$, we have $F(z)=O(z^{-2})$, and so, by Cauchy's theorem, the sum of the residues of $F(z)$ at all its poles is equal to zero. It is therefore evident that $f(r)$ is equal to the residue of $-F(z)$ at the origin.

By writing $-F(z)$ in the form

$$-F(z)=\frac{r}{z^2}\left(1-\frac{z}{r}\right)^{r-1}\left(1+\frac{(r-1)z}{r}\right)\left[\left(1-\frac{z}{1}\right)\left(1-\frac{z}{2}\right)\ldots\left(1-\frac{z}{r}\right)\right]^{-1},$$

we see that this residue is equal to

$$r\left(-\frac{r-1}{r}+\frac{r-1}{r}+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r}\right);$$

and we have therefore proved that

$$f(r)=r\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r}\right).$$

This same expression for $f(r)$ is derivable from the second of Maunsell's expressions by applying Cauchy's theorem to the function

$$-\frac{r!\,(r-z)^r}{r^{r-1}z^2(1-z)(2-z)\ldots(r-z)}.$$

It follows that $f(r)$ is expressible in terms of the logarithmic derivate of the gamma-function by the formula

$$f(r)=r\psi(r+1)-r\psi(1),$$

and so it can be computed with the help of a table of $\psi(x)$, such as is given by H. T. Davis, *Tables of the Higher Mathematical Functions*, 1 (1933), 348-352.

The asymptotic expansion of $\psi(x)$ shows that

$$f(r)\sim r\log r-r\psi(1)+\tfrac{1}{2}-\sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{2nr^{2n-1}},$$

and consequently Maunsell's surmise that, for large values of r, $f(r)\sim r\log r$ is correct. The presence of the term in r in the asymptotic expansion of $f(r)$ means, however, that $r\log r$ is not a very good approximation to $f(r)$. G. N. WATSON.

1347. *Falling off a log.*

1. The method of calculating logarithms by proportional parts given on pp. 50-52 of the *Algebra Report* depends on the fact that $\log_{1+b}(1+bx)>x$, where $0<b<1$ and $0<x<1$. We can obtain a rough and ready upper bound by applying the same inequality to

$$\log_{1+b}\{(1+b)/(1+bx)\},$$

the proportional part of which is

$$\left(\frac{1+b}{1+bx}-1\right)\Big/(1+b-1),$$

or $$(1-x)/(1+bx),$$

and then $$\log_{1+b}(1+bx)<1-(1-x)/(1+bx)$$
$$=(1+b)x/(1+bx).$$

It is the object of this note to show that by using the arithmetic mean of these bounds we can calculate common logarithms correct to five decimal places from the table given on p. 51 of the *Report*.

2. To do this we require the leading terms in the expansion in powers of b of

$$\frac{f(x)}{x}\equiv\frac{1}{x}\log_{1+b}(1+bx)-\tfrac{1}{2}\left(1+\frac{1+b}{1+bx}\right)$$
$$=(1-\tfrac{1}{2}bx+\tfrac{1}{3}b^2x^2-\tfrac{1}{4}b^3x^3)(1+\tfrac{1}{2}b-\tfrac{1}{12}b^2+\tfrac{1}{24}b^3)$$
$$-\tfrac{1}{2}-\tfrac{1}{2}(1+b)(1-bx+b^2x^2-b^3x^3)$$
$$=-\tfrac{1}{12}b^2(1-3x+2x^2)+\tfrac{1}{24}b^3(1+x-8x^2+6x^3);$$

so that $$f(x)=-\tfrac{1}{12}b^2x(1-x)(1-2x)+\tfrac{1}{24}b^3x(1-x)(1+2x-6x^2)$$
$$=(-\tfrac{1}{12}b^2+\tfrac{1}{24}b^3)x(1-x)(1-2x)+\tfrac{1}{6}b^3x^2(1-x)^2$$
$$-\tfrac{1}{12}b^3x^3(1-x)^2$$
$$<(-\tfrac{1}{12}b^2+\tfrac{1}{24}b^3)x(1-x)(1-2x)+\tfrac{1}{96}b^3.$$

Now $$\tfrac{1}{10}-x(1-x)(1-2x)=\tfrac{1}{10}\{1-10x+30x^2-20x^3\}$$
$$=\tfrac{1}{10}\{(1-x)(1-9x+21x^2)+x^3\},$$

which is positive. Hence when $0<x<\frac{1}{2}$,

$$0<x(1-x)(1-2x)<\tfrac{1}{10},$$

and when $\frac{1}{2}<x<1$

$$-\tfrac{1}{10}<x(1-x)(1-2x)<0.$$

Hence the greatest numerical value of $f(x)$ is less than

$$(\tfrac{1}{12}b^2-\tfrac{1}{24}b^3)\tfrac{1}{10}+\tfrac{1}{96}b^3.$$

Putting $b=\frac{1}{10}$ and noting that $\log_{1\cdot1}10>24$, the greatest possible error in calculating common logarithms from the mean of the bounds is less than

$$\tfrac{1}{24}[\tfrac{1}{1200}\cdot\tfrac{19}{20}\cdot\tfrac{1}{10}+\tfrac{1}{96000}]$$
$$=\tfrac{1}{24}\cdot\tfrac{43}{480000}=\tfrac{43}{11520000}<4\cdot10^{-6},$$

so that we obtain accuracy to five decimal places.

3. In using the rule of § 1 we must be careful to calculate the proportional parts to as many figures as the table of powers of 1·1 allows: for example, for $\log_{1\cdot1}10$ the proportional part is

$$(10-9{\cdot}84973)\div{\cdot}984973={\cdot}15256,$$

and from the rule we then get $\log_{1\cdot1} 10 = 24\cdot15893$, which is ·00007 in excess of the true value.

The working for $\log_{10} 2\cdot7183$ is as follows:

The proportional part is

$$(2\cdot7183 - 2\cdot59374) \div \cdot259374 = \cdot48023.$$

The upper bound is

$$(\cdot48023 + \cdot04802) \div 1\cdot048023 = \cdot50405.$$

The mean of these is ·49214, and then

$$10\cdot49214 \div 24\cdot15893 = \cdot434297,$$

which, although only ·43430 is claimed, is actually correct to six decimal places. This abnormal accuracy is accounted for by the fact that $f(x) \equiv 0$ when $x = \{\sqrt{(1+b)} - 1\}/b$; and for $b = \frac{1}{10}$ this is $10(\sqrt{1\cdot1} - 1)$ or ·488. In the present case ·48023 is near to ·488. Also when $x = \frac{1}{2} - \epsilon$,

$$x(1-x)(1-2x) = (\tfrac{1}{4} - \epsilon^2) \,.\, 2\epsilon < \tfrac{1}{2}\epsilon.$$

Here $\epsilon < \cdot02$, and so $x(1-x)(1-2x) < \cdot01$, so that the error in the common logarithm can not exceed $4 \,.\, 10^{-7}$, accounting for the additional accuracy.

The point is further illustrated by $\log (1\cdot024)^2$, or 1·048576, the proportional part of which, namely, ·48576, is even nearer to ·488. Applying the rule we eventually get

$$\log_{10} 2 = \cdot30103000.$$

N. M. Gibbins.

1348. *The identity in Note* 1310 (XXII, p. 380).

If $\begin{bmatrix} n \\ r \end{bmatrix}$ denotes $\dfrac{n(n+1)\ldots(n+r-1)}{r!}$, and s_n denotes the finite sum

$$(\alpha^n + \beta^n) + \begin{bmatrix} n \\ 1 \end{bmatrix}(\alpha^{n-1} + \beta^{n-1})\frac{\alpha\beta}{\alpha+\beta} + \begin{bmatrix} n \\ 2 \end{bmatrix}(\alpha^{n-2} + \beta^{n-2})\left(\frac{\alpha\beta}{\alpha+\beta}\right) + \ldots,$$

then s_n is the coefficient of t^{n-1} in the product

$$\left(\frac{\alpha}{1-\alpha t} + \frac{\beta}{1-\beta t}\right)\left(1 - \frac{\alpha\beta t}{\alpha+\beta}\right)^{-n}.$$

Since identically

$$1 = \{(\alpha+\beta)t - \alpha\beta t^2\} + (1-\alpha t)(1-\beta t),$$

this product is equal to

$$(\alpha+\beta)t\left(\frac{\alpha}{1-\alpha t} + \frac{\beta}{1-\beta t}\right)\left(1 - \frac{\alpha\beta t}{\alpha+\beta}\right)^{-(n-1)} + \{(\alpha+\beta) - 2\alpha\beta t\}\left(1 - \frac{\alpha\beta t}{\alpha+\beta}\right)^{-n}.$$

But since

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} \equiv \frac{n(n+1)\ldots(2n-2)}{(n-1)!} = 2\begin{bmatrix} n \\ n-2 \end{bmatrix},$$

the coefficient of t^{n-1} in

$$\left(1-\frac{2\alpha\beta t}{\alpha+\beta}\right)\left(1-\frac{\alpha\beta t}{\alpha+\beta}\right)^{-n}$$

is zero, and s_n is $(\alpha+\beta)$ times the coefficient of t^{n-2} in the product

$$\left(\frac{\alpha}{1-\alpha t}+\frac{\beta}{1-\beta t}\right)\left(1-\frac{\alpha\beta t}{\alpha+\beta}\right)^{-(n-1)}.$$

That is, $s_n=(\alpha+\beta)s_{n-1}=(\alpha+\beta)^n$, since $s_1=(\alpha+\beta)$. E. H. N.

1349. *The identity in Note* 1310.

As in the original note we observe that $s(\phi)+s(\psi)$ is the coefficient of x^{-1} in the expansion of

$$\frac{1}{[x(1-x)]^n}\left\{\frac{\phi}{1-\phi x}+\frac{\psi}{1-\psi x}\right\}.$$

Setting $\phi+\psi=\phi\psi=\theta$, this expression may be written as

$$\frac{\theta(\overline{1-x}-x)}{[x(1-x)]^n}\cdot\frac{1}{1-\theta x(1-x)},$$

and we may clearly consider instead the modified expression

$$\frac{\theta(\overline{1-x}-x)}{[x(1-x)]^n}\left\{\frac{1-[\theta x(1-x)]^n}{1-\theta x(1-x)}\right\}$$

or

$$\left\{\frac{\theta}{x^n(1-x)^{n-1}}-\frac{\theta}{x^{n-1}(1-x)^n}\right\}\sum_{r=0}^{n-1}\theta^r x^r(1-x)^r,$$

which may be written

$$\sum_{r=0}^{n-1}\frac{\theta^{n-r}}{x^{r+1}(1-x)^r}-\sum_{r=0}^{n-1}\frac{\theta^{n-r}}{x^r(1-x)^{r+1}}.$$

Now, if $r\geqslant 1$, the coefficient of x^{-1} in

$$\frac{1}{x^{r+1}(1-x)^r}-\frac{1}{x^r(1-x)^{r+1}} \quad \text{is} \quad \begin{bmatrix} r \\ r \end{bmatrix}-\begin{bmatrix} r+1 \\ r-1 \end{bmatrix},$$

which is zero. The only term yielding x^{-1} is thus θ^n/x and

$$s(\phi)+s(\psi)=\theta^n.$$

J. L. BURCHNALL.

1350. *A note on symmetric functions.*

In the Theory of Equations a symmetric function of $x_1, x_2, \ldots, x_n$ is often evaluated by expressing it in terms of quantities :

$$[p]\equiv x_1{}^p+x_2{}^p+\ldots+x_n{}^p.$$

I do not remember having seen the general method of carrying out this process in the books, so perhaps it may be worth while giving it.

Suppose that the symmetric function is $\Sigma x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_m^{\epsilon_m}$ $(m \leqslant n)$. Let

$$m = p_1 a_1 + p_2 a_2 + \dots + p_r a_r, \quad p_1 + p_2 + \dots + p_r = k, \quad \dots\dots\dots(1)$$

where the p's and a's are positive non-zero integers, no two of $a_1, a_2, \dots, a_r$ being equal. Divide $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ into k sets, p_1 of which contain a_1 of the ϵ's, p_2 contain $a_2, \dots, p_r$ contain a_r. This can be done in s ways, where

$$m! = s(p_1!)(p_2!) \dots (p_r!)(a_1!)^{p_1}(a_2!)^{p_2} \dots (a_r!)^{p_r}.$$

Let the sums of the ϵ's in each set be $c_1, c_2, \dots, c_k$ respectively. Then add the products $[c_1][c_2] \dots [c_k]$ for each of the s possible ways of dividing the ϵ's into k sets and multiply the result by

$$(-1)^{m+k}\{(a_1-1)!\}^{p_1}\{(a_2-1)!\}^{p_2} \dots \{(a_r-1)!\}^{p_r}.$$

Finally, sum the expressions so obtained for every possible way of putting m into the form given by (1). The sum is equal to $\Sigma x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_m^{\epsilon_m}$.

We have assumed here that no two of $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ are equal. If the ϵ's are equal in sets of $p, q, \dots$, we must divide the sum by $p!\ q! \dots$. (P. A. MacMahon, *Combinatory Analysis*, Cambridge University Press, 1915, Vol. I, p. 6, gives the case $\epsilon_1 = \epsilon_2 = \dots = \epsilon_m = 1$.)

A proof by induction of the above statement is obtained on forming the product $\Sigma x_1^{\epsilon_{m+1}} \Sigma x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_m^{\epsilon_m}$.

On putting $x_1, x_2, \dots, x_n$ all equal we have

$$n(n-1)(n-2) \dots (n-m+1) = \Sigma(-1)^{m+k} \frac{m!}{p_1! p_2! \dots p_r!\, a_1^{p_1} a_2^{p_2} \dots a_r^{p_r}} n^k,$$

where the summation Σ extends over all values of the a's and p's satisfying (1). H. HILTON.

1351. *On the direction angles of a straight line in a circumscriptible polygon.*

It is well known that, in a triangle $A_1A_2A_3$, having as angles A_1, A_2, A_3, a straight line Δ cuts the sides under angles $\theta_{2,3}, \theta_{3,1}, \theta_{1,2}$ such that

$$\sin A_1 \cos \theta_{2,3} + \sin A_2 \cos \theta_{3,1} + \sin A_3 \cos \theta_{1,2} = 0.$$

I have also given in *Mathesis*, 1930, p. 378, the relation

$$\sin 2A_1 \cos 2\theta_{2,3} + \sin 2A_2 \cos 2\theta_{3,1} + \sin 2A_3 \cos 2\theta_{1,2} = 0.$$

The similarity of the two formulas suggests the following generalisation.

1. Let Γ be the unit-circle with centre O and radius 1, and T the point on Γ corresponding to the *turn* t,* the unit-point Ω corresponding to the turn 1 ; then, if θ is the angle $(O\Omega, OT)$,

$$t = \cos\theta + i \sin\theta.$$

Two points T and T' corresponding to the turns τ and τ' being

* F. and F. V. Morley, *Inversive Geometry*, 1933, p. 15.

given on Γ, the point T_0 corresponding to the turn $\tau\tau'$ is derived from the fact that the triangle $T'OT_0$ is directly similar to ΩOT. Hence ΩT_0 is parallel to TT'.

Conversely, the point T_1, corresponding to the turn τ/τ', is such that $T'T_1$ is parallel to ΩT.

2. Let now $A_i (i = 1, 2, \dots, n)$ be n points on Γ corresponding to the turns t_i and, Δ being the direction perpendicular to $O\Omega$, denote by $2\alpha_{i,i+1}$ and $\theta_{i,i+1}$ the angles *

$$(OA_i, OA_{i+1}), \quad (A_iA_{i+1}, \Delta).$$

The parallel through A_i to ΩA_{i+1} cuts Γ at the point B_i corresponding to the turn t_{i+1}/t_i; the mid-point B_i' of the arc ΩB_i corresponds to $(t_{i+1}/t_i)^{\frac{1}{2}}$ and the angle $\Omega OB_i'$ equals $\alpha_{i,i+1}$; hence

$$(t_{i+1}/t_i)^{\frac{1}{2}} = \cos \alpha_{i,i+1} + i \sin \alpha_{i,i+1},$$
$$(t_{i+1}/t_i)^{-\frac{1}{2}} = \cos \alpha_{i,i+1} - i \sin \alpha_{i,i+1},$$

and, m being any integer,

$$\sin m\alpha_{i,i+1} = \frac{1}{2i} (t_i^{-m/2} t_{i+1}^{m/2} - t_i^{m/2} t_{i+1}^{-m/2}).$$

3. The parallel through Ω to A_iA_{i+1} cuts Γ at the point C_i corresponding to the turn $t_i t_{i+1}$, and it follows that the point C_i' corresponding to $(t_i t_{i+1})^{\frac{1}{2}}$ is the mid-point of the arc A_iA_{i+1}; therefore the angle $\Omega OC_i'$ equals $\theta_{i,i+1}$; hence

$$(t_i t_{i+1})^{\frac{1}{2}} = \cos \theta_{i,i+1} + i \sin \theta_{i,i+1},$$
$$(t_i t_{i+1})^{-\frac{1}{2}} = \cos \theta_{i,i+1} - i \sin \theta_{i,i+1},$$

and

$$\sin m\theta_{i,i+1} = \frac{1}{2i} (t_i^{m/2} t_{i+1}^{m/2} - t_i^{-m/2} t_{i+1}^{-m/2}),$$
$$\cos m\theta_{i,i+1} = \tfrac{1}{2} (t_i^{m/2} t_{i+1}^{m/2} + t_i^{-m/2} t_{i+1}^{-m/2}).$$

4. Using the foregoing expressions, we obtain the following general theorem:

If a polygon $A_1A_2 \dots A_n$ is inscribed in a circle, the angles at the centre corresponding to the sides A_iA_{i+1} being denoted by $2\alpha_{i,i+1}$, and if any straight line Δ cuts the sides A_iA_{i+1} under the angles $\theta_{i,i+1}$, then, m being any integer,

$$\Sigma \sin m\alpha_{i,i+1} \sin m\theta_{i,i+1} = 0,$$
$$\Sigma \sin m\alpha_{i,i+1} \cos m\theta_{i,i+1} = 0.$$

5. In the case of a triangle $A_1A_2A_3$, the angles A_1, A_2, A_3 of the triangle are equal to $\alpha_{2,3}$, $\alpha_{3,1}$, $\alpha_{1,2}$, and

$$\Sigma \sin mA_1 \sin m\theta_{2,3} = 0,$$
$$\Sigma \sin mA_1 \cos m\theta_{2,3} = 0.$$

6. When $\alpha_{1,2} = \alpha_{2,3} = \dots = \alpha_{n-1,n} = \alpha,$

then $\alpha_{n,1} = \pi - (n-1)\alpha,$

* $A_{n+1} \equiv A_1$, $\alpha_{n,n+1} \equiv \alpha_{n,1}$, $\theta_{n,n+1} \equiv \theta_{n,1}$.

and $\theta_{1,2}=\theta,\quad \theta_{2,3}=\theta+2\alpha,\ \dots,\quad \theta_{n-1,n}=\theta+2(n-2)\alpha,$

$$\theta_{n,1}=\theta+2(n-2)\alpha+[\pi-(n-2)\alpha].$$

Hence $$\sin\alpha\{\sin\theta+\sin(\theta+2\alpha)+\dots+\sin[\theta+2(n-2)\alpha]\}$$

$$=-\sin[\pi-(n-1)\alpha]\sin[\pi+\theta+(n-2)\alpha],$$

or $$\sin\theta+\sin(\theta+2\alpha)+\dots+\sin[\theta+2(n-2)\alpha]$$

$$=\frac{\sin(n-1)\alpha}{\sin\alpha}\sin[\theta+(n-2)\alpha];$$

this is a well-known formula ; the similar expression for the sum of the cosines of the terms of an arithmetic progression will be obtained in the same way.

7. This special case suggests the following form for the first general relation :

If the sum of the angles

$$\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n$$

equals π, *then,* θ *being any angle,*

$$\begin{aligned}&\sin m\alpha_1\sin m\theta\\ &+\sin m\alpha_2\sin m(\theta+\alpha_1+\alpha_2)\\ &+\sin m\alpha_3\sin m(\theta+\alpha_1+2\alpha_2+\alpha_3)\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+\sin m\alpha_n\sin m(\theta+\alpha_1+2\alpha_2+2\alpha_3+\dots+2\alpha_{n+1}+\alpha_n)=0\end{aligned}$$

the second general formula may also be written in a similar form.

R. GOORMAGHTIGH.

1352. *A curious rectangle.*

Although what follows does not give a simple proof of the property considered in Note 1261, *Gazette*, XXI, p. 412, it will perhaps be interesting to complete the Notes 1293, XXII, p. 287, and 1294, XXII, p. 288, by a generalisation of the theorem.

The general problem to find two circles touching three given circles, with centres A, B, C, tangent to each other, when ABC is *any* triangle, has been solved before ; * but the fact that, when, as in the present case, ABC is a *rectangular* triangle, the centre of one of the obtained circles forms *always* a rectangle with A, B and C, does not seem to have been mentioned yet. Also the expressions of the corresponding radii in value of those of the given circles are then very simple.

* Aubert, *Nouvelles Annales de Mathématiques*, 1876, p. 318 ; Lemoine, *Nouvelle Correspondance Mathématique*, 1880, p. 513 ; Neuberg, *Mathesis*, 1906, p. 59.

In *Mathesis*, 1932, p. 445, I have extended the here mentioned properties to the case of circles cutting under the same angle the three given circles.

Let ABC be any triangle, D, E, F and D_a, E_a, F_a the contact points of the sides with the incircle and with the circle exscribed in the angle A ; let further be a, b, c the sides, $2s = a+b+c$, and r, r_a, r_b, r_c the radii of the in- and exscribed circles. Circles tangent to each other and having their centres at A, B, C are the circles $\Gamma_a(A, s-a)$, $\Gamma_b(B, s-b)$, $\Gamma_c(C, s-c)$ or $\Gamma_a'(A, s)$, $\Gamma_b'(B, s-c)$, $\Gamma_c'(C, s-b)$, and the two other similar groups.

It is easy to prove by inversion that, if the circles $\Gamma(\omega, \rho)$ and $\Gamma'(\omega', \rho')$ are tangent to the first triad, *the barycentric coordinates of ω and ω' in ABC are as*

$$r_a \mp a, \quad r_b \mp b, \quad r_c \mp c,$$

and the expressions of ρ, ρ' are given by

$$\rho^{-1}, \quad \rho'^{-1} = 2r^{-1} \mp [(s-a)^{-1} + (s-b)^{-1} + (s-c)^{-1}]. \quad \ldots\ldots\ldots(1)$$

Similarly, if the circles $\Gamma_1(\omega_1, \rho_1)$, $\Gamma_1'(\omega_1', \rho_1')$ touching the triad $(\Gamma_a', \Gamma_b', \Gamma_c')$ are considered, *the barycentric coordinates of ω_1 and ω_1' are as*

$$-(r \pm a), \quad r_c \pm b, \quad r_b \pm c,$$

and

$$\rho_1^{-1}, \quad \rho_1'^{-1} = \pm 2r_a^{-1} - s^{-1} + (s-b)^{-1} + (s-c)^{-1}. \quad \ldots\ldots\ldots(2)$$

If then ABC is rectangular at A,

$$r_a + r_b = \tfrac{1}{2}bc[(s-a)^{-1} + (s-b)^{-1}] = 2bc^2/[c^2 - (a-b)^2] = a+b,$$

$$r_b - r_c = \tfrac{1}{2}bc[(s-b)^{-1} - (s-c)^{-1}] = 2bc(b-c)/[a^2 - (b-c)^2] = b-c,$$

$$r - r_b = \tfrac{1}{2}bc[s^{-1} - (s-b)^{-1}] = -2b^2c/[(a+c)^2 - b^2] = -a+c.$$

It follows that

$$-(r_a - a) = r_b - b = r_c - c,$$

$$r + a = r_c + b = r_b + c.$$

Therefore, *when the triangle ABC formed by the centres of the given circles is rectangular at A, the centres ω and ω_1 coincide and form a rectangle with A, B and C.*

If A' is the image of A through the mid-point of BC, the obtained property will be easily *verified*, if it is noted that $AA' = a$, $A'B = b$, $A'C = c$; this proves also that the corresponding radii ρ and ρ_1 are s and $s-a$, or, in value of the radii α, β, γ and α', β', γ' of Γ_a, Γ_b, Γ_c and Γ_a', Γ_b', Γ_c',

$$\rho = \alpha + \beta + \gamma, \quad \rho_1 = \alpha' - \beta' - \gamma'.$$

Further, (1) and (2) show also that

$$\rho'^{-1} = 3\alpha^{-1} + \beta^{-1} + \gamma^{-1}, \quad \rho_1'^{-1} = -3\alpha'^{-1} + \beta'^{-1} + \gamma'^{-1}.$$

In the case of the triangle, 5, 4, 3, $\alpha = 1$, $\beta = 3$, $\gamma = 2$ and $\rho = 6$; $\alpha' = 6$, $\beta' = 2$, $\gamma' = 3$ and $\rho_1 = 1$. The centre ω' has as barycentric coordinates 11, 7, 5 and $\rho' = 6/23$; ω_1' is the image of B through A and $\rho_1' = 3$.

R. GOORMAGHTIGH.

1353. *A Note on Projective Geometry.*

The first year university student who has learnt projective geometry from the current school textbooks, which invariably employ metrical methods, finds he must abandon most of what he

has learnt before, and start projective geometry over again as an axiomatic system. After mastering many difficult new ideas about homographic ranges, he then finds that in the actual demonstrations of fundamental theorems a relatively vague, so-called "abbreviated" argument, appealing to the uniqueness of certain constructions, takes the place of the actual determination of (1, 1) related ranges. Instead of these fundamental theorems illustrating the basic technique of the subject they lead the student to suspect that his previous labour was simply wasted. The object of this short note is to show for the following three fundamental theorems that the detailed construction of the mechanism of the relevant homographic ranges is short and simple. It is hoped that these examples will serve to give the student a clearer idea of the importance of the theory of related ranges in the projective geometry of the conic.

The theorem that if a conic Σ is defined in relation to pencils through two points A and B, and if C is a point on Σ, then Σ is determined also by (1, 1) related pencils through B and C, is of fundamental importance and merits a demonstration in which the relatedness of the pencils through B and C is explicitly determined. So, too, Desargues' theorem that a pencil of conics meets a line in point pairs in involution. The third theorem which we shall consider, that if (P) and (P') are related ranges of points on a conic then (PP') envelops a conic, seems to be invariably proved by the "abbreviated argument".

Let a conic Σ be determined by related pencils through G and K. Let f be the join of G, K and let l, through G, correspond to f, through K, and n, through K, to f, through G. Suppose that A and P are other points on Σ. If GA and KP meet at D, GP and KA at E and l and n at H, then by the dual of Pappus' theorem, D, E, H are collinear. Let PA and DE meet GK at C and L, and CD meet HG and HK at S and T respectively.

Then since $H\{GL\,KC\} = -1$, we have $\{SD\,TC\} = -1$ and so GT is the harmonic conjugate of GH with respect to GA and GK. Thus T is independent of P.

Now KP, through K, and AP, through A, are both in perspective with TD, through T, and so KP and AP are corresponding rays of related pencils through K and A.

$\alpha, \beta, \gamma, \delta$ are the common points of a pencil of conics. To show that the conics meet a line l in point pairs in involution. Let $\gamma\alpha, \gamma\beta, \delta\alpha, \delta\beta$ meet l at A, B, A', B', and let F and S be points not on l and such that A, F, S are collinear. Let P be a point on l, and suppose that PS and $A'F$ meet at R, $B'R$ and BS meet at H, FH and l meet at P'. Then (P) on l is in perspective with (R) on $A'F$ from S, (R) is in perspective with (H) on BS from B', and (H) is in perspective with (P') on l from F. Thus (P) and (P') are (1, 1) related. Let $A'F$ and $P'S$ meet at V; FH, PR at N; and let $B'V$ meet PR and SH at X and W respectively.

Then $$S\{VXWB'\} = S\{P'PBB'\} = H\{NPSR\}$$
$$= F\{P'PAA'\} = S\{VRFA'\};$$

therefore P, F, W are collinear.

Thus (P') on l is in perspective with (V) on $A'F$ from S, (V) is in perspective with W on SB from B', and (W) is in perspective with (P) on l from F. The ranges (P) and (P') are therefore in involution.

A, A' and P, P' are corresponding members of a pair of (1, 1) related ranges on a conic Σ. U and V are the double points of the ranges, so that AP' and PA' meet UV at L, say. The tangents at U and V meet at T; AA', PP' meet at O; UA, VP' at S; OA, TU at M; OP', TV at N; AA', TV at M'; and OS, UV at R.

The polar of L passes through O and S. The polar of T passes through L, so that the polar of L passes through O, S and T. Hence, by applying Desargues' theorem to the perspective triangles UTV, AOP', it follows that L, M, N are collinear. Since $\{LURV\} = -1$, VM and UN intersect on ST. By considering the Brianchon hexagon T, U, M, O, N, V, it follows that AA', TU, TV, PP' touch a conic through U and V. Thus PP' touches the unique conic determined by the related ranges on TU, TV, in which M corresponds to M', U on TU to T on TV, and V on TV to T on TU.

R. L. GOODSTEIN.

1354. *Apollonian numbers.*

The first of the interesting formulae for Apollonian numbers given by Dr. C. Dudley Langford in Note 1289 in the May *Gazette*, is, I find, a particular case of a more general formula,

$$[(2n+2)a+b]^2+[2na+(2n+1)b]^2 \equiv 2[(2n+1)a+nb]^2 + 2[a+(n+1)b]^2. \quad \text{.....(i)}$$

Putting $a = b = 1$, we find

$$(2n+3)^2+(4n+1)^2 \equiv 2(3n+1)^2+2(n+2)^2. \quad \text{.....(ii)}$$

This formula clearly gives a triangle for any positive integral or fractional value of n.

There is obviously no limit to the number of such formulae which can be found by substituting numerical values for a and b.

Putting $a = 1$, $b = 2$, we obtain

$$(2n+4)^2+(6n+2)^2 \equiv 2(4n+1)^2+2(2n+3)^2,$$

which might have been found by multiplying throughout by 2 in (ii). In fact, any of these formulae may be " doubled " in this way.

R. HAMILTON DICK.

1355. *π-Formulae.*

Having heard that π has been calculated to 707 decimal places, and having seen some of the relatively slowly convergent formulae with which this remarkable feat is stated to have been accomplished, it seems to me that an enormous amount of time and labour would have been saved had more convergent formulae been developed beforehand. Here are two which I evolved by tabulating the

solutions in positive integers of the equations $\frac{x \pm y}{xy \mp 1} = \frac{m}{n}$.

Writing $[m]$ for arc $\tan \frac{1}{m}$,

$$\frac{\pi}{4} = 17[99] + 17[162] + 39[200] + 42[242] + 34[343] + 15[500] + 12[924] - 3[1568]$$
$$= 56[200] + 76[242] - 17[243] + 68[343] + 32[500] + 12[924] - 20[1568] - 17[28800] - 17[29282].$$

It will be observed that the greatest prime factor of any denominator is eleven.

Of course, these formulae are of interest nowadays only as mathematical curiosities. R. HAMILTON DICK.

1356. *On Note* 1248 : *To draw a square of which a given parallelogram is the orthogonal projection.*

A short time before his death, Mr. Alfred Lodge sent me an interesting completion of the construction published in the *Gazette* for October 1937, p. 284, Fig. 2.

The note gives no reason for the position of XY, the line of intersection of the two planes, in Fig. 2 [where by the way the angle ABC should have been obtuse as in Fig. 1].

Mr. Lodge gave the following easy construction. "Complete the diameter FOF' ; then the arc CF' subtends the angle θ at F. Therefore cut off the arc AG = the arc CF' along the arc AC. Then BG is the direction of the XY line."

The somewhat elaborate construction for E earlier in the note may be replaced by the following, which is more easily proved :

"If BD is the diagonal of the parallelogram, take E in AC so that the angles EBC and ABD are equal." B. E. LAWRENCE.

1357. "*Why all this fuss about method ?*"

From a script :

Solve
$$2\sqrt{x} = x - 3,$$
$$\therefore \quad 2x^{\frac{1}{2}} = x - 3,$$
$$\therefore \quad (x^{\frac{1}{2}})^{\frac{1}{2}} = x - 3,$$
$$\therefore \quad x^{\frac{1}{2}} = x - 3,$$
$$\therefore \quad x^{\frac{1}{2}} - x^{\frac{1}{2}} = 3,$$
$$\therefore \quad x^{\frac{1}{2}} = 3,$$
$$\therefore \quad x = 3 \times 3 = 9.$$ G. L. P.

1358. *The solution of a quartic equation.*

The roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ .....(1)

are the points of intersection of the parabola $y = x^2$ and the conic

$$y^2 + axy + \lambda x^2 + cx + (b - \lambda)y + d = 0, \quad \text{.....(2)}$$

where λ is an arbitrary parameter. Now if we choose λ so that (2) is a pair of straight lines, then (2) can be written in the form

$$(y+\alpha x+\beta)(y+\gamma x+\delta)=0,$$

and hence (1) can be written in the form

$$(x^2+\alpha x+\beta)(x^2+\gamma x+\delta)=0,$$

whence (1) can be solved.

The required condition for λ is

$$\lambda d+2\,.\,\tfrac{1}{2}a\,.\,\tfrac{1}{2}c\,.\,\tfrac{1}{2}(b-\lambda)-\lambda\{\tfrac{1}{2}(b-\lambda)\}^2-(\tfrac{1}{2}c)^2-d(\tfrac{1}{2}a)^2=0,$$

which simplifies to

$$\lambda^3-2b\lambda^2+\lambda(ac+b^2-4d)+da^2-abc+c^2=0.$$

Thus, as in the other methods, the solution depends on the solution of a cubic equation, but this method has the slight advantage that because most people remember the condition for two straight lines, the cubic can be written down without having to do any elimination. H. V. LOWRY.

1359. *A note on the construction of numerical examples of algebraic integrals of the form*

$$\int\frac{(fx+g)dx}{(ax^2+2bx+c)\sqrt{ax^2+2\beta x+\gamma}}=\int\frac{(fx+g)dx}{X\sqrt{X_1}}.$$

It is assumed that the first trinomial in the denominator has no real factors, as otherwise the rational part of the integrand may be broken up into partial fractions, so that two simple standard forms result.

The usual method of attack on integrals of this type is to use the substitution

$$y^2=\frac{\alpha x^2+2\beta x+\gamma}{ax^2+2bx+c}=\frac{X_1}{X}.$$

From this we have

$$2y\frac{dy}{dx}=\frac{(ax^2+2bx+c)(2\alpha x+2\beta)-(\alpha x^2+2\beta x+\gamma)(2ax+2b)}{X^2},$$

whence $X_1^{\frac{1}{2}}X^{\frac{3}{2}}\dfrac{dy}{dx}=(\alpha b-a\beta)x^2+(\alpha c-a\gamma)x+\beta c-b\gamma=E$, say.

In constructing numerical examples of the integral, it is necessary that E should have rational factors. E can be written

$$a\alpha\left\{\left(\frac{b}{a}-\frac{\beta}{\alpha}\right)x^2+\left(\frac{c}{a}-\frac{\gamma}{\alpha}\right)x+\frac{\beta}{\alpha}\cdot\frac{c}{a}-\frac{b}{a}\cdot\frac{\gamma}{\alpha}\right\},$$

or

$$a\alpha\{(l-\lambda)x^2+(m-\mu)x+\lambda m-l\mu\},$$

where

$$\frac{b}{a}=l,\quad\frac{c}{a}=m,\quad\frac{\beta}{\alpha}=\lambda,\quad\frac{\gamma}{\alpha}=\mu.$$

Thus E will break up into rational factors if

$$(m-\mu)^2-4(l-\lambda)(\lambda m-l\mu)=F, \text{ say,}$$

is a perfect square. This will be achieved if we can choose l, m, λ, μ so that

$$\mu-m=p+q, \quad \ldots\ldots(1)$$

$$\lambda-l=p, \quad \ldots\ldots(2)$$

$$l\mu-\lambda m=q, \quad \ldots\ldots(3)$$

where p and q are unequal but otherwise arbitrary, and in this case F will be equal to $(p-q)^2$.

The equations (1), (2) and (3) contain six unknowns, so that three may be chosen arbitrarily; it will be found that p, q and m are convenient for arbitrary choice, as (1) then gives μ and (2) and (3) become simultaneous equations for l and λ.

E.g. choose $p=2$, $q=3$, $m=2$.

Then from (1), $\mu=7$, and (2) and (3) become

$$\lambda-l=2,$$

$$7l-2\lambda=3.$$

These give $l=\frac{7}{5}$, $\lambda=\frac{17}{5}$.

Based on these results, here follows a worked example of the type.

To evaluate

$$I=\int\frac{(4x+5)\,dx}{(5x^2+14x+10)\sqrt{(5x^2+34x+35)}}=\int\frac{(4x+5)\,dx}{X\sqrt{X_1}}.$$

Put $$y^2=\frac{5x^2+34x+35}{5x^2+14x+10}=\frac{X_1}{X}.$$

Then

$$2y\frac{dy}{dx}=\frac{(5x^2+14x+10)(10x+34)-(5x^2+34x+35)(10x+14)}{X^2};$$

Thus $$X_1^{\frac{1}{2}}X^{\frac{3}{2}}\frac{dy}{dx}=-25(2x^2+5x+3)$$

$$=-25(2x+3)(x+1).$$

Stationary values of y^2 as a function of x are given by $x=-\frac{3}{2}$ and $x=-1$. These stationary values are

$$x=-\tfrac{3}{2}: \quad y^2=\frac{5\cdot\frac{9}{4}-34\cdot\frac{3}{2}+35}{5\cdot\frac{9}{4}-14\cdot\frac{3}{2}+10}=-19;$$

$$x=-1: \quad y^2=\frac{5-34+35}{5-14+10}=+6.$$

Hence

$$6-y^2=\frac{30x^2+84x+60-5x^2-34x-35}{X}$$

$$=\frac{25x^2+50x+25}{X}=\frac{25(x+1)^2}{X}. \quad \ldots\ldots(A)$$

Also $$y^2+19=\frac{5x^2+34x^2+35+95x^2+266x+190}{X}$$

$$=\frac{100x^2+300x+225}{X}$$

$$=\frac{25(2x+3)^2}{X}. \quad \text{.....(B)}$$

Now if we substitute for dx in I, we get

$$I=\int\frac{(4x+5)\,dx}{X\sqrt{X_1}}=\int\frac{(4x+5)X_1^{\frac{1}{2}}X^{\frac{3}{2}}\,dy}{X\sqrt{X_1}(-25)(2x+3)(x+1)}$$

$$=-\frac{1}{25}\int\left\{\frac{2}{2x+3}+\frac{1}{x+1}\right\}X^{\frac{1}{2}}\,dy$$

$$=-\frac{2}{25}\int\frac{X^{\frac{1}{2}}}{(2x+3)}\,dy-\frac{1}{25}\int\frac{X^{\frac{1}{2}}}{x+1}\,dy.$$

Now from (A) and (B),

$$\frac{X^{\frac{1}{2}}}{x+1}=\frac{5}{\sqrt{6-y^2}} \quad \text{and} \quad \frac{X_1^{\frac{1}{2}}}{2x+3}=\frac{5}{\sqrt{y^2+19}}.$$

Thus $$I=-\frac{2}{5}\int\frac{dy}{\sqrt{y^2+19}}-\frac{1}{5}\int\frac{dy}{\sqrt{6-y^2}}$$

$$=-\frac{2}{5}\sinh^{-1}\frac{y}{\sqrt{19}}-\frac{1}{5}\sin^{-1}\frac{y}{\sqrt{6}}+c$$

$$=-\frac{2}{5}\sinh^{-1}\left\{\frac{5x^2+34x+35}{19(5x^2+14x+10)}\right\}^{\frac{1}{2}}$$

$$-\frac{1}{5}\sin^{-1}\left\{\frac{5x^2+34x+35}{6(5x^2+14x+10)}\right\}^{\frac{1}{2}}+c.$$

I have assumed that this method of integration is well known, although I personally am indebted, both for the method and for the proof of its validity, upon which I have not touched, to Mr. A. S. Ramsey, of Magdalene College, Cambridge. R. SIBSON.

1360. *On the three-cusped hypocycloid.*

The tangential equation of the three-cusped hypocycloid, discussed in Note 1266 (*Gazette*, 1937, p. 418), is obtained very easily by writing that any tangent and its reciprocal line, *i.e.* the straight lines having as barycentric equations

$$\lambda x+\mu y+\nu z=0, \quad \mu\nu x+\nu\lambda y+\lambda\mu z=0,$$

must be perpendicular; hence, by the well-known orthogonality condition,

$$\lambda\mu\nu\Sigma a^2-\Sigma\lambda(\mu^2+\nu^2)bc\cos A=0.$$

This is the required equation. R. GOORMAGHTIGH.

CORRESPONDENCE.

"A WEIGHTY MATTER."

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—Mr. Grattan-Guinness has been so polite to me personally and has included so many shrewd observations in the superstructure of his article in the October *Gazette*, that it is with regret that I attempt to make him see the fallacy which underlies his argument.

By finding that two identical bodies balance, we may infer that the balance works truly, but *no inference extending our knowledge of the* (*ex hypothesi*) *identical bodies can be drawn*. Except therefore as a test of the balance, it seems undesirable that in our teaching we should either place identical bodies in the pans of a balance or suppose them to be so placed.

Mr. Grattan-Guinness says that the balance *compares* weights (equal weights, of course, but a lever would compare unequal weights) but must *measure* something, so it measures masses. It is as if one said "a ruler measures lengths, therefore it calculates the areas of rectangles". For this, the correct version is, "a ruler measures lengths, therefore it enables us to calculate the areas of rectangles, *because we know* the way in which these areas depend on lengths." In the same way, the balance enables us to measure masses, *because we know* or assume that bodies which have at the same place equal weights have also equal masses. But to know this, we must have *beforehand* some idea of the meaning of mass. How is this idea got?

It is *not* got by saying "mass is the property in which identical bodies agree when you have ceased to consider all the properties such as volume, colour, shape, smell, which you can detect by methods other than the use of the balance". No one knows how many different properties a given body possesses, and in any case this explanation is too negative to give any positive idea of mass. A boy does not get the idea of a straight line from the statement that not more than one of them passes through two points, but from something he can see, such as a line drawn in pencil, on which he can base a mental concept. Similarly he gets the idea of mass from one of two ways in which mass appeals to the senses.

The first is as "quantity of matter". The boy buys milk by the pint or butter by the pound and knowing that he does not drink volume or eat weight, he forms the idea of quantity of matter. The difficulties underlying this phrase are well known. Quantity of matter is the product of volume and density, while to explain density is to get round again to mass. Still the idea is a positive one and *one which the boy has got*. One can hear the teacher saying, "If there was only one kind of matter mass would be measured by volume but in some kinds of matter the atoms, particles, protons or what you will, are packed together more tightly than in other kinds of matter. If you could count these protons, etc., you could get a real direct measure of mass. As it is, you have to fall back on the assumption that equal masses have (at the same place) equal weights, and determine masses by weighing."

The second way in which the boy gets the idea of mass is from the difficulty of starting into motion bodies which are at rest and still more from the difficulty of stopping bodies which are in motion. This leads to the dynamical explanation of mass or inertia and does not, as far as I am aware, lead to arguments in a circle such as are involved in "Volume times density". Hence the statement that the inertia idea is the *only scientific* way of explaining mass as opposed to the vague and popular idea of quantity of matter, and hence also the suggestion that the word mass if used scientifically may be replaced by

inertia and that if it cannot be so replaced it is not being used scientifically but in one of its popular senses.

To turn to the later part of Mr. Grattan-Guinness' article. His suggestions of "lig" and "tug" are most attractive but unfortunately most unpractical, for it is extraordinarily difficult to get people to adopt these pet names. Teachers love the pet names which they invent themselves, but regard as undignified and unscientific those which other people invent. I was brought up on "velo" and "celo" and still think they have advantages over "ft./sec." and "ft./sec.^{2}" but I fear that "velo" and "celo" are dead. "Slug" survives in use by aircraft designers and that it does so speaks volumes for the forceful personality of its inventor or backer, Professor Perry, but the mere name "slug" has prevented generations of teachers from using the British engineer's unit of mass at all. If Mr. Grattan-Guinness or anyone else other than a dictator can introduce "lig" and "tug", good luck to him. He might begin by trying to get strong resolutions in favour of the idea from the various boards controlling the larger certificate examinations.

Yours truly,

C. O. TUCKEY.

THE NEED FOR REFORM IN THE TEACHING OF MATHEMATICS.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—I hope I may be afforded a little space for some remarks on Mr. Dockeray's article in the October *Gazette*. He gave admirable expression to my own misgivings about the status of mathematics in the schools. Quite apart from any future change in our economic system, envisaged by Mr. Dockeray, the present standing of the subject cannot be regarded as satisfactory.

In the first place, it is a fact that in the Sixth Forms of a large number of schools, mathematics, as a cultural subject, has little chance of existence because the only boys taking it are members of the Science Sixth, who all do physics or chemistry as well. No facilities exist for the mathematical specialist and in these circumstances, mathematics is mainly regarded as a necessary adjunct to the science subjects. As quite a few promising mathematicians are not at all attracted by the chemistry laboratory, they inevitably turn to arts subjects.

Also I cannot help feeling that, in the early stages, some of the modern methods of teaching geometry (for example, construction of models), unless very carefully employed by expert teachers of mathematics, tend to give the subject a materialistic bias from the outset. In trying to widen the appeal of the subject, are we not in danger of diminishing its aesthetic value?

Yours truly,

E. JOHNSON.

REFERENCE FOR SIMILAR TRIANGLES.

To the Editor of the *Mathematical Gazette*.

SIR,—The notation for references for similar triangles which Mr. Tuckey suggests on p. 486 of the December *Gazette* seems to me very artificial and entirely unnecessary. A.A.A., S.A.S., S.S.S. are quite enough without bracketing an S. (If he brackets one S, why does he not bracket every S?) There is no danger of confusing congruence and similarity. S.A.S. means only that we have used two sides and the included angle; whether it is for congruence or for similarity is clearly stated in the line at the end of which the S.A.S. is written.

Thus if two triangles *RST*, *XYZ* have been proved congruent by using the three-side case of congruence, the last line appears thus :

$$\therefore \triangle RST \equiv \triangle XYZ. \quad \text{(S.S.S.)}$$

If the triangles have $RS/XY = ST/YZ = TR/ZX$,
the last line is

$$\therefore \triangle RST \text{ and } \triangle XYZ \text{ are similar. (S.S.S.)}$$

In the same way, in

$$\triangle RST \equiv \triangle XYZ \quad \text{(S.A.S.)}$$

and $\triangle RST$ and $\triangle XYZ$ are similar. (S.A.S.)

S.A.S. is quite clear, and there is no possibility of doubt whether we are dealing with congruent or similar triangles. Again, "$\triangle RST$ and $\triangle XYZ$ are similar (A.A.A.)" is perfectly clear.

What would be very useful would be some accepted symbol for " is similar to ".

Yours truly,

A. W. SIDDONS.

To the Editor of the *Mathematical Gazette*.

SIR,—I am fully in agreement with Mr. Tuckey's letter in the December *Gazette*. My own reference symbols, however, are :

for congruence	*for similarity*
S A S	S A S :
A S A	A A :
A A S	S S S :
S S S	
R S S	
A S S (ambiguous)	

The sign for ratio (:) stands out more clearly if no full stops are used, so that we write

S A S , not S.A.S.,
S A S : , not S.A.S. : .

Yours truly,

L. LINES.

ON A REVIEW.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—We note in the review of our *Modern School Arithmetic* in the December *Gazette* the statement :

" In spite of this attention to the degree of accuracy, the authors obtain on p. 210 by the use of four-figure logarithm tables £115 15s. 3d. as the amount of £100 in 3 years at 5% compound interest."

This statement is quite untrue. In fact, we say on p. 210 :

" $\log A = \log 100 + 3 \log 1{\cdot}05$.

Using four-figure tables . . . $A = £115{\cdot}8$.

$\therefore$ Amount = £115 16s.

By direct calculation without logs, A = £115 15s. 3d. " ;

and we go on to point out that seven-figure tables would be necessary for practical calculation of interests by this method.

With regard to the remark that the mil method of decimalising money to three places is given without warning of a lurking danger, we would refer

F.C.B. to pp. 101, 102, where the exact error is indicated and an example is worked out showing the correction required when a sum of money, decimalised to three places, is multiplied by 1000.

Yours truly,

R. N. HAYGARTH,
E. V. SMITH.

To the Editor of the *Mathematical Gazette.*

DEAR SIR,—We notice with regret that in the review of our book *A School Arithmetic* we are stated " to give the mil method of decimalising money to three places without any warning of a lurking danger " ; in fact this danger is pointed out on page 157 of our book.

Yours faithfully,

JOHN S. CHANNON,
A. McL. SMITH.

To the Editor of the *Mathematical Gazette.*

DEAR MR. EDITOR,—I should like without hesitation to apologise to the authors of *A School Arithmetic* (Messrs. Channon and McLeish Smith) and of a *Modern School Arithmetic* (Messrs. Haygarth and Smith) for some passages which I wrote in reviewing their books. They are :

(i) Both give the mil method of decimalising money to 3 places without any warning of a lurking danger.

(ii) The authors of the *Modern Arithmetic* obtain by the use of 4-figure logs £115 15s. 3d. as the amount of £100 in 3 years at 5% C.I.

With regard to (i) : in the *School Arithmetic* a warning is inserted at the end of the chapter, and in the *Modern Arithmetic* the Exercises on 3-place work are followed by methods for obtaining the full decimal with exercises on them.

With regard to (ii) : the authors get £115 16s. by 4-figure logs, and add, " By direct calculation without logs A = £115 15s. 3d."

Yours sincerely,

F. C. BOON

1250. The minimum of national celibacy (ascertained by dividing the number of males in the community by the number of females and taking the quotient as the number of wives or husbands permitted to each person) is secured in England (where the quotient is 1) by the institution of monogamy.—G. B. Shaw, *The Revolutionist's Handbook.* [Per Mr. G. P. Rawlings.]

1251. (Suggested by Gleaning 1225). The father of a new student when bringing him to the University, after calling to see the Professor, drew his assistant on one side and besought him to tell him what his son must do that he might stand well with the Professor. " You want your son to stand weel with the Profeessorr? " asked MacFarlane. " Yes." " Weel, then, he must just have a guid bellyful o' mathematics "!—S. P. Thompson, *Life of Lord Kelvin*, p. 420. [Per Professor G. N. Watson.]

1252. For the harmony of the World is made manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty. A greater than Verhaeren had this in mind when he told of ' the golden compasses, prepared in God's eternal store '. Sir D'Arcy Thompson, *Growth and Form.* [Per Mr. A. F. Mackenzie.]

REVIEWS.

Lehrbuch der Gruppentheorie. I. By H. ZASSENHAUS. Pp. vi, 151. RM. 13; geb. RM. 13.75. 1937. Hamburger Mathematische Einzelschriften, 21. (Teubner)

Anyone opening this slender book expecting merely to find an account of the more elementary and standard parts of the subject will be speedily disillusioned. Although the volume before us is only the first part of an extended treatise, it already takes the reader from the foundations right through to some of the most important advances of the last few years, results which now find their place in a textbook for the first time. To have compressed so much material, both new and old, into so small a space is indeed a remarkable feat. And it may be that the economy of the style will make some parts of the book not altogether easy reading. But probably the main credit for this high "specific content" is due to the systematic grouping of the exposition around the central notion of a homomorphic correspondence. This is an arrangement which not only lends the book a close-knit logical structure: it also serves to correct the impression which group theory sometimes gives of being a sequence of beautiful theorems having little to do with one another. Such an impression could scarcely survive a reading of this volume. Both as regards originality of treatment and thorough modernity of outlook, this promises to be when complete a contribution of first-rate importance to the literature of the subject, and one which all serious students of groups will wish to possess.

A *homomorphic correspondence* between two groups is a function $f(x)$ whose argument x ranges over the elements of one group G, and whose values $f(x)$ lie in another group G' and satisfy the functional relation $f(xy)=f(x)f(y)$. The set of values $f(x)$ then also forms a group $f(G)$ which may or may not coincide with G'. If the correspondence is (1, 1) it is called an *isomorphism*; G and $f(G)$ may then be regarded as different manifestations of one and the same abstract group. If in addition we suppose $G=f(G)$, we arrive at the fundamental notion of the *automorphisms* of a group. The ideas of self-conjugate sub-group and quotient group, which proved so essential in the earlier history of the subject, are also immediate corollaries of the notion of homomorphic correspondence. And in taking this as his guiding principle, Herr Zassenhaus has I think definitely succeeded in imposing a certain logical order on the wide variety of problems which are treated in his first volume.

The book is a little difficult to criticise on grounds of omission, as no indication has been given of the plan of the remaining volume or volumes. But I understand that representation theory and the theory of the defining relations of a group—both notable absentees from Vol. I—fall due for treatment in Vol. II. Incidentally, these two very important branches of the subject derive quite naturally from the idea of homomorphic correspondence and so fit neatly into the plan of the present work. Representation theory as at present developed may be defined as the study of the case in which G' is the group of all homogeneous linear transformations in a certain number of variables; while relation theory on the other hand is the study of the case in which G is a free group and $f(G)=G'$ is the group whose defining relations are in question.

The present volume, however,—apart from a section on permutational representations—is not concerned with such matters; indeed free groups are not even mentioned. It begins with a somewhat miscellaneous chapter on axioms, permutations, congruences, cyclic and Platonic groups, conjugates and normalizers. The central core of the book is the long and important second chapter, which covers the general theory of homomorphic correspond-

ences. The treatment is based on the notion of a " group with operators ", an *operator* of a group G being simply the case $G'=G$ of a homomorphic correspondence. It is to the introduction of this notion, largely due to Krull and Emmy Noether, that the modern form of group theory owes its great generality as compared with the older versions. As will be familiar to any reader of van der Waerden's *Moderne Algebra*, the various types of algebraic system which have proved of most interest and importance, such as fields, rings, linear spaces, algebras, etc., all enter as particular cases under this concept of a " group with operators ". It is this fact which gives the fundamental Jordan-Hölder-Schreier theorem its great range of applications. This theorem is here given in its best modern form, the proof being based on a notable paper by Zassenhaus himself. The chapter also contains a great number of other important results which cannot be mentioned here. The statement on p. 42, however, that " the problem of the existence of outer automorphisms leaving every class of conjugate elements invariant is still unsolved " is not correct. This question was settled in the affirmative by Burnside, *Proc. London Math. Soc.* (2) 11 (1913), 40.

Chapter 3 deals with the construction problem : how to build up the more complicated groups from their simple constituents. It covers the theory of direct products based on the work of Fitting, and the theory of extensions due to Schreier. Elementary divisor theory and the basis theorem for Abelian groups with a finite number of generators find their natural place at this point. Much of this chapter represents advances of the last decade, and I think that a rather more leisurely treatment would have made for easier reading ; in particular, § 2 on the Remak-Schmidt theorem seems too condensed.

Chapter 4 deals for the most part with older results of a less general character, viz. Sylow's theorem and the properties of prime-power groups ; though even here we find much that has not before found its way into the textbook literature. This is particularly welcome in view of a fact which is becoming increasingly clear, that so many of the main problems about finite groups reduce in the last resort to questions of prime-power group theory.

Finally, in the concluding chapter of the volume, the new and fruitful idea of the *Verlagerung* (or impression) of a group on one of its sub-groups is introduced. This idea is contained implicitly in some famous work of Burnside on monomial representations. It is exploited here, in a most brilliant chapter, to prove the classical theorems of Burnside just referred to, the very important generalisations which have lately been made by O. Grün, and the celebrated *Hauptidealsatz* of algebraic number theory.

Criticism might be directed at the scanty number of references to the literature. It is a little surprising, for instance, to find Wedderburn's well-known theorem on finite division algebras with no mention of Wedderburn's name. The reader is referred to Burkhardt's encyclopedia article for the older papers. But that article only extends to the first years of the century. This leaves a big gap, for nearly all the papers listed for reference at the end of the book were published within the last ten years.

However, these points become trivial when compared with the positive merits of the book. Herr Zassenhaus has rendered a most valuable service to all who are interested in this subject, and we shall await the publication of the later volumes with impatience. P. HALL.

Einführung in die neueren Methoden der Differentialgeometrie. II. By J. A. SCHOUTEN and D. J. STRUIK. Second edition. Pp. xii, 338. RM. 16 ; geb. RM. 17.50. 1938. (Noordhoff, Groningen)

The work before us is the second part of the second edition of a book

which was published in one volume some years ago; but the plan has been so altered and extended that this edition is really an entirely new book. The volume under consideration is the work of D. J. Struik. It is devoted entirely to geometry, the mathematical analysis employed having been developed by J. A. Schouten in the first volume, which appeared in 1935. (See *Math. Gazette*, Vol. XX, p. 60.) Though Riemannian geometry occupies a prominent place in the discussion, the treatment embraces also linearly-connected (or affine) spaces L_n. The geometrical objects introduced are not necessarily real, neither are the coordinates employed. A Riemannian space is denoted as usual by V_n; and, if the metric is positive definite, the author describes the space as "ordinary".

The volume embraces four chapters, of which the first is devoted to curves. This begins by considering curves in Euclidean 3-space, finding the equivalent of Frenet's formulae in any system of coordinates; and then extends the results to real or imaginary curves in Riemannian space of n dimensions. Points of inflection of any order are discussed, and the theorem is proved that an ordinary real curve in a V_n is uniquely determined when the positions of its tangent and normals are known at one point, and its $n-1$ curvatures are known functions of the arc-length measured from that point. An application of preceding results is then made in considering evolutes of any order of real curves in Euclidean space of n dimensions. A section is devoted to the treatment of curves in a linearly-connected space L_n, results being found which are analogous to the formulae of Frenet, though not exactly generalisations of them; and Hlavaty's theorem on the determination of a curve by its affine curvatures is established. Another section—the longest of the chapter—discusses congruences of curves, chiefly in Riemannian space. The usual topics arise including geodesic and normal congruences, orthogonal ennuples, Ricci's coefficients of rotation, congruences canonical with respect to a given congruence, and rectilinear congruences in Euclidean n-space. The chapter closes with a discussion of the properties of a system of paths (*bahnsystem*), *i.e.* a system of curves such that, through two specified points in a given region, passes one and only one curve of the system. The chief emphasis is given to a "natural" system of paths, *i.e.* one which can be transformed into a system of geodesics by a conformal transformation of the fundamental tensor.

Chapter II is devoted to hypersurfaces of a Riemannian space. It covers the usual ground, beginning with the first and second fundamental tensors for the hypersurface, and then passing on to proofs of the theorems of Meusnier, Euler and Dupin. In connection with normal curvature, geodesics and asymptotic lines of the hypersurface are considered; and the discussion then proceeds to conjugate directions, lines of curvature and principal curvatures of the hypersurface at any point, and to asymptotic lines of any order. The chapter concludes after the equations of Gauss and Codazzi have been established, and their special forms for a hypersurface of a flat space R_n.

Chapter III deals at some length with a general sub-space V_m of a Riemannian V_n. By considering the curvature of a curve in V_m relative to the enveloping space, the author is led to the discussion of normal curvature and curvature relative to V_m, and thence to geodesics of V_m and the definition of geodesic sub-spaces. The mean curvature vector of V_m is then introduced, leading to the mention of minimal sub-spaces and some of their properties. The discussion then turns to congruences of curves in a sub-space, and in particular to conjugate systems and asymptotic systems. As sub-spaces of some importance, surfaces (V_2) are considered in some detail, and their properties at axial, planar and spatial points are examined. Special mention is given to Steiner surfaces and Veronese surfaces. The author then dis-

cusses curvatures of any order of a sub-space of V_n, and proves the Frenet formulae for the sub-space. The equations of Gauss, Codazzi and Ricci are also established for the sub-space, and a number of applications are made. Some attention is given to questions concerning small deformations of a sub-space V_m, and also to the different kinds of scalar curvature of the sub-space. The last part of the chapter deals fairly fully with sub-spaces of a space of constant curvature; and in this connection the class of a Riemannian space is discussed.

Chapter IV, the longest in the book, is devoted to a number of selected topics, among which the linearly-connected space L_n occupies a prominent place. The chapter opens with a discussion of various curvature properties of an L_n, and proves the Gauss-Codazzi-Ricci equations for an $L_n{}^m$. It then passes to small deformations of a manifold, among which infinitestimal deformations which leave the metric of a space unaltered are given prominence. A fairly lengthy section discusses transformations of an affine space A_n of symmetric connections, such that the paths of A_n correspond to those of the transformed space, and proves among others the theorems that two Riemannian spaces S_n of constant curvature can be so transformed into each other, or an S_n into a flat space R_n; while a V_n can be so transformed into an S_n only when the former is also an S_n. This section closes with a discussion of projective connections, which are the basis of projective differential geometry. The author then proceeds to consider the conformal representation of one Riemannian space upon another. Incidentally it is shown that, for a conformal transformation of the fundamental tensor of a V_n, umbilics of the space remain umbilics, and the principal directions of a hypersurface of the space are invariant. A good deal of attention is devoted to conformal-euclidean Riemannian spaces, *i.e.* spaces which may be conformally transformed into flat spaces. The discussion is also extended to affine spaces by the consideration of conformal connections; and the chapter closes with a fairly long section dealing with Hermitian connections.

A large number of important and useful examples are given throughout the text; and at the close of the book about fifty pages are devoted to hints and methods for their solution. Many of the examples are really additional theorems. There is a bibliography of twenty-two pages which, though not intended as a complete one, will prove very useful to the reader. An extensive index of sixteen pages should also prove a great convenience. The dedication is given very appropriately to Dr. Tullio Levi-Cività, who is one of the outstanding figures in this field of work.

A book, written by two men who have contributed so much to the development of the subject, is clearly a valuable addition to its literature. As the treatment is by no means elementary, the reader is advised to acquire some knowledge of Riemannian geometry from a more introductory work* before beginning the study of this one. The mathematician who wishes to extend his knowledge of Differential Geometry, and is prepared to take the trouble necessary to master the notation employed by the writers, will find these two volumes a mine of information and, we hope, a source of inspiration. To such a one we confidently recommend the book.

C. E. WEATHERBURN.

Tafels van e^x. By H. W. HOLTAPPEL. Pp. xxxi, 132. *Fl.* 6.00. 1938. (Noordhoff, Gröningen)

The reviewer of mathematical tables has not the sinecure that some may think he has; if it is considered that the primary object of a review is to enable possible users or purchasers to decide whether the tables are suitable

* *e.g.* the reviewer's *Riemannian Geometry* (Cambridge, 1938).

for his purpose or not, then the reviewer should indicate not only the contents and mode of presentation, but also the degree of reliability of the tables. When tables are produced by authors of experience and known reliability, little difficulty is encountered, but when the tables are by an "amateur" about whom the reviewer has no previous knowledge, the only way is the laborious one of examining samples of the tables for errors; when, as in this case, grave suspicion of carelessness is aroused in one direction, but not confirmed by considerable checking in other tables, the reviewer, whose samples are necessarily small, cannot complete his task.

Putting aside the question of accuracy for the time being, the book of tables under review consists of a praiseworthy attempt to tabulate e^x and e^{-x} in an accessible form, thus filling a badly felt gap in this field. The author in his preface, which is translated from the Dutch into English, French, German and Italian, calls attention to this deficiency, and in doing so stresses the difficulty of finding references to other published tables; after an elaborate search of libraries and numerous references to others who might be expected to know of the existence of such tables, he refers only to the tables of Newman and Glaisher in *Transactions of the Cambridge Philosophical Society*, and then only to those published in 1883, apparently being unaware of the table of e^x published in the same place by Newman in 1889, and of the fundamental tables of Van Orstrand in *Memoirs of the National Academy of Sciences* (1921). In this respect, the author deserves sympathy, as all these tables are difficult of access and no comprehensive bibliography exists.

In the present tables, the main tabulation consists of e^x and e^{-x} to 10 decimals in the range 0·000 (0·001) 10·000. In addition to several short tables giving e^x and e^{-x} to from 16 to 25 decimals for integral and fractional values of x, e^x only is tabulated for the following ranges and with the number of decimals indicated in square brackets.

Range 0·0(0·1)9·9 for x[18], and for $10^{-2}x$[24].

Range 0·000(0·001)0·999 for x[17], $10^{-3}x$ [20], $10^{-6}x$[20], $10^{-9}x$[20] and $10^{-12}x$[20].

The last five tables should prove particularly useful, as they enable direct or inverse interpolation to be carried through three figures at a time. It is not quite clear, however, why the first of these tables should be given to three decimals less than the others, though the number of figures given is ample for practically all purposes.

Although new style figures, without heads or tails, are used throughout, and the paper upon which the tables are printed is not of a high quality, these points do not detract greatly from the general appearance of the volume, which is on the whole well arranged and printed.

Numerous examples, mainly of inverse interpolation, are given; each of these is designed to show the use of the tables in different cases according to the number of decimals or significant figures required in the result. The general method of determining x from a given value of e^x is by successive applications of the approximate formula

$$(x - x_0) = (e^x - e^{x_0})e^{-x_0}.$$

By studying the auxiliary tables provided, the user should be able to develop fairly simple methods in all the cases he requires. With a function like e^x, methods of interpolation and tabulation are bound to depend upon the individual requirements of the user, and no tabulation can be entirely satisfactory from all points of view. Although e^{-x} is obtainable immediately from e^x, one cannot help feeling that the main table, in which both functions are given to the same number of decimals, is not balanced; it is thought that,

both from the point of view of general utility and ease of interpolation, an increase in the number of decimals in e^{-x} would have been justified. The author acknowledges that the table of e^{-x} has been formed by rounding off Newman's values, and so its merit lies purely in its accessibility.

To return to the most important question of accuracy, the reviewer discovered, in checking the few fundamental values given to about 20 decimals, no fewer than 20 errors on one page; none of these was obviously an error of proof-reading, but about half were small rounding-off errors and the remainder fundamental errors of calculation in e^x for $x=12(1)24$. These latter errors are systematic and correspond to an error of about 4 in the 29th decimal of e, or of the omission of the later terms in the series. From the point of view of the main tables, these errors are unimportant, but they show a lack of appreciation of the essentials of computing. This impression is deepened by the fact that the author points out in his preface that these values differ from those of Glaisher, to whom he attributes the errors. It is peculiar that later he says "For the positive values of x I found to be in agreement with Mr. Glaisher's table". Several (about 10 in all) rounding-off errors have been found in the auxiliary tables, though a complete examination has not been made; these appear to be accidental. In view of this tendency to error, samples of the main table of e^x were differenced to detect errors in proof-reading; only one such error was found. Apart from the numerical accuracy of the tables, there are several small errors in the examples; it is particularly illuminating to read "Lust figure uncertain" at the foot of one example!

Taking into account the errors and slips, there is no doubt that these tables will be of great use to all who require values of the exponential function to more than five or six figures; the accuracy is not all it should be, but only extensive use or special investigation can decide this point. If, as appears probable, this is the author's first adventure into the field of table making, he deserves congratulation for an estimable piece of work in a particularly difficult field. D. H. S.

Theoretical Hydrodynamics. By L. M. MILNE-THOMSON. Pp. xxviii, 552. 31s. 6d. 1938. (Macmillan)

The publication of a book on Hydrodynamics, that seeks to present the modernised version of a subject so enormously grown in stature during the past two decades, is something of an event in mathematical-physical circles. Lamb's *Hydrodynamics*, a book of comparable magnitude, has served during forty years as the Bible on this subject to all serious students; but it has been clear for many years now, that the content of the subject had too completely outgrown the form of that work to make it possible by any simple alteration of text to cope with the necessary modern demand. Nothing short of a complete re-writing of the whole subject could possibly have availed, and Milne-Thomson is to be congratulated on having undertaken this very necessary task.

The early history of hydrodynamical theory dates from the same period as the studies in electrical flow that were pursued in the latter part of the nineteenth century, and much of the early treatment was little more than a redressing of electrical theory in hydrodynamical garb. Indeed it would scarcely have been necessary to change many of the terms, for the phraseology of electricity—terms like flow, current and resistance—were themselves drawn from hydrodynamical images. But theory does not develop out of theory alone, and the practical needs of aerial flight and of naval design quickly brought into being such an enormous body of experimental knowledge that the old electrical treatment could no longer be regarded as satisfactory. Conditions of surface drag; streamlining and the origin of eddying in the

wake of a body ; the critical changes in the patterns of fluid motion ; and the state of turbulence in which that simple pattern appears to be obliterated—these all began to crystallize out as features of hydrodynamic flow that were foreign to orthodox theory. If mathematical physics was to play its part as a guide in experimental design, then the theory of hydrodynamics required a complete overhauling. Lamb's *Hydrodynamics* grew up in the older atmosphere and lived through with remarkable endurance into the new. In estimating, therefore, how far Milne-Thomson has really faced the modern demand, we must see this work as an attempt to re-erect hydrodynamics for survival in the turbulent atmosphere of today.

The new treatment shows itself in a variety of ways. The book begins with a series of plates, beautifully executed, showing the nature of the eddying flow past fixed bodies. Throughout the work there are frequent references to experimental findings in aero-dynamics, and much of the mathematical analysis seeks to explain the detailed effects that have been detected in wind-tunnel tests. In this respect Milne-Thomson has certainly struck a very modern note. In many other ways, too, he makes a break with the older treatment. Vector methods, for example, are adopted in those sections of the subject for which they are particularly suitable without prejudice to others for which they are not ; and with this I feel in especially strong agreement. In all, he devotes sixty-four pages, more than a tenth of the book, to the study of vectors and the treatment of the complex variable, and these sections are so excellently done that they are well worth reading for their own sake. They are included in chapters 2 and 5, in which in illustration he also derives a number of the general hydrodynamical theorems.

The general properties of fluid motion, the formulae for pressure, energy and vorticity and the dynamical equations that govern their change are covered by chapter 3, while chapters 4 and 6 to 14 inclusive form a complete unit in which the principles of conformal mapping are applied to two-dimensional motion. It is in these sections that it becomes apparent with what skill the older complex theory, applicable to a perfect fluid, has been applied in the newer hydrodynamics. The subject is extended again in chapter 18 to three-dimensional motion to cover the case of aeroplane wings of finite span, involving an examination of the theory of trailing vortices. The groundwork for this is, however, prepared in chapter 13, where the author provides a discussion of the steady motion and stability of the street of vortices that follow in the wake of a moving two-dimensional body. The last chapter is devoted to a discussion of the viscous fluid with a brief analysis of the theory of the boundary layer. There are in all 508 exercises and 330 diagrams, the latter a very welcome improvement on Lamb's unpictorial text.

To appreciate the difference in stress between the treatment by Lamb and that by Milne-Thomson, it is worth while contrasting the omissions of the one against the inclusions of the other. In the present work tidal oscillations on a rotating globe are completely ignored. Lamb devotes 32 pages to its treatment. Very much the same situation can be seen in relation to the oscillations of a spherical mass of liquid. Milne-Thomson, on the other hand, deals more fully with the conditions in the wake of a moving body, and, while restricting himself always to the theory, succeeds in keeping the subject nevertheless very close to the problems of practice. Thus while Milne-Thomson succeeds in creating a modern aero-dynamic flavour, Lamb is geo-physical.

On the other hand, there is one remarkable difference in which Lamb is, in a sense, more modern than Milne-Thomson. The subject of viscosity is handled by the latter in the last chapter, almost as if it had been added as an afterthought, when the author had already tired of his colossal task. Lamb devoted

three times this amount to this subject. It is evident that the treatment of circulation for the purpose of achieving lift in an aerofoil, the separation of the fluid into a boundary layer in which the movement is viscous but slow, and the rest of the fluid in which the motion is that of one devoid of viscosity—these are merely devices for avoiding the fundamental difficulty of treating the non-linear equations of viscous flow. Great as has been their ingenuity and remarkable their success, the fact does remain that in future further inroads will require to be made into the treatment of viscous motion without these simplifying and therefore restrictive conditions. It would seem desirable, therefore, that the problems that have so far yielded to solution should receive considerable prominence. G. I. Taylor's treatment of the stability of motion in the fluid between two rotating cylinders is accorded little more than a mere mention while, as far as I can discover, turbulence and eddy-viscosity are neither in the index nor in the book. Even boundary layer theory might well have merited a fuller treatment.

There is one further criticism that must be mentioned, to decide how far Milne-Thomson has really built well and securely on the new foundations. The motion of propeller blades and of shells have raised a series of problems of first importance in relation to air resistance at speeds in the neighbourhood of sound. Modern high speed aircraft designed for record-breaking are already approaching speeds at which the effects of the compressibility of the air may shortly make themselves manifest. A considerable volume of research in this field has already been published both in this country and abroad during recent years, and much careful experimental work has been carried through in the new high pressure wind-tunnels. It seems a pity that matters, such as these, growing points of the subject, should have been omitted ; but this may, of course, have been due to the fact that the author probably had in mind the present generation of honours students whose studies in subjects such as these tend to lag considerably behind the front line of advance.

These are not intended as complaints against what is a first rate book but as suggestions regarding what might be incorporated in successive additions. For of one thing, at any rate, we can be certain, that the new Lamb, however it may be roasted, is one of the most palatable dishes it has been my pleasure to consume. The book is beautifully printed and the diagrams are attractively executed. It will be bought—as it richly deserves to be—by every serious worker in this subject. H. L.

Lehrbuch der Mathematik zum Selbstunterricht und für Studierende der Naturwissenschaften und der Technik. By G. SCHEFFERS. 7th Edition. Pp. 743 ; 438 figures. RM. 15 (bound). 1938. (W. de Gruyter, Berlin)

The book is extremely well printed and at first glance it seems surprising that a publication of this size can be sold so cheaply. In the preface, however, the author explains that the book has been a best seller for thirty-three years and that this edition differs from the previous one only in the elimination of a few slips and misprints. The expense involved in the setting up of new type was therefore avoided. Further, in the opinion of Professor Scheffers, there is no reason to introduce any changes.

The obvious popularity of this German textbook on elementary mathematics makes it of interest to those engaged in teaching the elements of mathematical method and logic.

To say that it is a book on elementary mathematics may give a wrong impression, for the book is very different from those to which this description is normally applied. One point that strikes the reader immediately is that much of the book, possibly one-third of it, contains the detailed working out of large numbers of examples, but there are no examples upon which an ambitious

student might try his skill and test his growing powers of mathematical manipulation. Everything is explained in great detail, but not at all heavily.

The book is written for those who use mathematics as a tool in science and technology. Its aim is to assume as little as possible and to take a beginner from a very low rung on the mathematical ladder and raise him gently to such a position that he will be able to appreciate the more difficult mathematical work that might appear in research or advanced mathematical literature. This level is so delightfully vague that it is impossible to say whether or not the aim of the author is realised. It can be said without hesitation, however, that those parts of elementary mathematics which are described are explained extremely well and lucidly.

Dr. Scheffers is principally concerned with elementary calculus and algebra. Chapter 1 is concerned with mensuration and the idea of a function. Absolute error, relative error and rounding-off are carefully explained. Constants, variables and functions are dealt with, but no mention is made of discontinuous functions. Chapter 2 is devoted to the "meaning" of linear functions, quadratic functions, bounding values, limits, infinitesimals and differential coefficients. The following chapter treats of polynomials, factors, maxima and minima of polynomials, solution of quadratics, differentials of function of polynomials and the inversion of a function. Chapter 4, headed "A Few Points from Analytical Geometry", deals with the very elementary parts of straight line, circle, ellipse, hyperbola, oblique axes, asymptotes and trilinear coordinates. Chapter 5 discusses the fundamentals of integration, functions with the same differential coefficient, quadrature and applications to mean values, linear and quadratic moments. Chapter 6 contains a better introduction to logarithms and the slide rule than one usually finds in a book on elementary mathematics. Surprisingly, however, the name "Napier" is not mentioned at all.

After Chapter 6 the author strays from the more or less narrow path described above and wanders up by-paths for very short distances. In Chapter 7 he talks about the exponential function, polar coordinates, logarithmic spirals, hyperbolic functions and the catenary. Chapter 8 introduces the circular functions, addition formulae and periodic functions. In the following chapter higher differential coefficients, maxima and minima, curvature, evolutes, involutes, motion along straight and curved lines, and the cycloid are grouped together. In Chapter 10 mean value theorems, Lagrange's interpolation formula and Taylor's theorem follow each other. The succeeding chapter evaluates a few standard integrals and introduces complex numbers and Fourier series. Chapter 12 then discusses functions of more than one variable, partial differentiation, analytical geometry of the plane and double integration.

The choice of subjects and their grouping are rather strange to those accustomed to more or less systematic mathematics even in its elementary aspects. One must, however, accept the fact that the book is an extremely popular one. The most that one could say about a similar book written in English would be that it would be of some use to those who teach elementary pure mathematics. L. R.

The Fifty-nine Icosahedra. By H. S. M. COXETER, P. DU VAL, H. T. FLATHER, and J. F. PETRIE. 26 pp. + 20 plates. Paper cover, 4s. 6d. 1938. University of Toronto Studies, Mathematical Series, No. 6. (University of Toronto Press; Humphrey Milford)

This pamphlet describes the application of the process of *stellation* to the regular solids. By stellation is meant the derivation of one polyhedron from another by extension of the faces of the original until they meet other such extensions, to form new faces, edges, and vertices. The definition is modified

so as to include cases in which a " face " of the derived solid consists of several isolated portions all, of course, in the same plane. This process evidently leaves the number of such faces unaltered. In the paper reviewed here, restrictive rules have been formulated which mean, in effect, that the final polyhedron must retain the full symmetry of the parent regular solid, except possibly for reflexive symmetry. Thus the rules allow a solid which is not identical with its reflection ; such a solid is called, rather unhappily, *unselfreflexible*. If identical with its reflection, the solid is called *selfreflexible*.

It may clarify the process if we describe the plane polygons obtained by stellating a regular pentagon. If we extend two non-adjacent sides until they meet, we add an isosceles triangle to the pentagon, giving a stellation of little symmetry. If this is done in all possible ways, the result is a five-pointed star, or *pentagram*. If, again, we omit the original pentagon, we obtain a set of five isosceles triangles which are " vertex-connected " and arranged so that the full pentagonal symmetry is retained. The restrictive rules would be such as to exclude all stellations except the two latter, on the ground of insufficient symmetry. Further, the three-dimensional rules are framed to exclude solids which can be built up from two or more fully symmetrical stellations ; a similar two-dimensional rule would exclude the collection of five isosceles triangles, since the boundary is a combination of those of the pentagon and pentagram combined. Thus the pentagram remains as the only allowable stellation of the pentagon.

The method is applied in the paper to the octahedron, dodecahedron and icosahedron ; there are no finite stellations of the tetrahedron or cube. The octahedron gives only the " stella octangula ", which consists of two interpenetrating tetrahedra. From the dodecahedron are derived the four regular star polyhedra of Kepler and Poinsot. Only in the case of the icosahedron can new forms be obtained by stellation of a regular polyhedron ; it is shown that there are 32 selfreflexible and 27 unselfreflexible icosahedra, though they are not all given for the first time in the present paper. Two methods of classification are used ; by means of possible bounding faces in section 2 and according to the sets of solid cells from which the polyhedron is built up in section 3.

The 59 resulting icosahedra are illustrated, together with a bounding face of each, in the 20 plates. The illustrations, a magnificent piece of work, are beautifully reproduced from drawings ; they are in white against a black background with the edges of the solid in black. The average diameter of an illustration is about two inches. Section 4 of the paper contains notes on the plates and supplementary descriptions of the solids. It is perhaps of interest to note here that the usual conception of " a solid " fails to apply to some of the polyhedra ; for instance, one of them, unique in this respect, consists of twelve completely separate portions. Others are vertex-connected, (*i.e.* they consist of separate solid cells, two or more of which may share a common vertex), or edge-connected. One solid, in particular, is of interest, although its peculiarity is not pointed out in the paper ; this, also unique, is **e, f, g,** on plate V, and has a multiply-connected volume as its interior. Many of these polyhedra have holes right through them, and in some cases, for clarity, only half the polyhedron has been drawn.

The reasoning used in the derivation and description is clear and fairly elementary. Quite a number of the solids provide excellent opportunities for making simple models, whilst others, for example those which are merely vertex-connected, require all the tricks which a model-maker can devise. Although no mention of this is made in the paper, I believe I am right in saying that models of all the icosahedra have been constructed by Flather.

J. C. P. MILLER.

Les nouvelles méthodes de la théorie du potentiel et le problème généralisé de Dirichlet. By C. DE LA VALLÉE POUSSIN. Pp. 46. 15 fr. 1937. Actualités scientifiques et industrielles, 578 ; publications de l'Institut Mathématique de l'Université de Strasbourg, II. (Hermann, Paris)

La notion de capacité. By F. VASILESCO. Pp. 49. 15 fr. 1937. Actualités scientifiques et industrielles, 571 ; exposés sur la théorie des fonctions, X. (Hermann, Paris)

Much progress has been made recently in potential theory by the use of Stieltjes integrals and the concept of " capacity ", a generalisation to the theory of sets of points of the familiar idea of electrostatic capacity. This progress was described by Professor de la Vallée Poussin in his lectures at University College, London, a year ago, which were printed in the *Gazette*, February 1938. In the first of the tracts under review, Vallée Poussin gives much the same results as he gave in his London lectures, with some variations of the methods employed. In the second tract, the idea of generalised capacity, treated briefly in the first tract, is discussed in full. T. A. A. B.

Sur les valeurs exceptionnelles des fonctions méromorphes et de leurs dérivées. By G. VALIRON. Pp. 53. 18 fr. 1937. Actualités scientifiques et industrielles, 570. Exposés sur la théorie des fonctions, IX. (Hermann, Paris)

One of the earliest results on derivatives and exceptional values is a theorem due to Borel to the effect that there is no integral function $f(z)$ (other than a polynomial) such that the equations

$$f(z)=a, \quad f^{(\nu)}(z)=b, \quad b\neq 0, \quad \nu\geqslant 1,$$

have only a finite number of roots. This theorem, like many others, has an analogue in the theory of normal families from which it can be deduced, viz. : *A family of functions $f(z)$ which are regular in D and such that $f(z)\neq 0, f^{(\nu)}(z)\neq 1$, $(\nu\geqslant 1)$ is normal in D.* This tract is concerned with the latter theorem, which is due to Miranda, and with other similar results of a more complicated nature covering meromorphic as well as regular functions.

The proofs are naturally very much condensed, and a good deal is assumed ; but the references should be sufficient for anyone with a little knowledge of the general theory of integral functions. A simple proof of Schottky's theorem is included ; but the constants are not the best possible. The later results in the tract are obtained by the author's direct method for proving Picard's theorem. M. L. CARTWRIGHT.

A New Geometry. By A. W. SIDDONS and K. S. SNELL. Pp. xvi, 302. 3s. 6d. 1938. Separately published, **Notes and Answers to the Exercises.** Pp. 16. 1s. 1938. (Cambridge University Press)

This is an elementary textbook based on the well-known Geometries of Godfrey and Siddons, but entirely re-written, re-arranged and modified in the light of *A Second Report on the Teaching of Geometry in Schools* of the Mathematical Association. It is designed to follow a short preliminary course and complete it up to the requirements of the School Certificate Examinations.

The book is arranged in three sections: Part I, pp. 154, the irreducible minimum; Part II, pp. 155 to 231, desirable additions; Part III, pp. 232 to 300, systematic course, followed by an index covering two pages. The theorems of the Systematic Course, numbered and grouped, are enunciated in order at the beginning of the book on pages xi to xvi, and each theorem always carries its own special number.

Part I (chapters 1 to 15) covers rather more than three-quarters of the whole course. Here the emphasis is on the acquisition of knowledge of geometrical

facts and the power to use them. Very gradually, and almost imperceptibly, more and more emphasis is transferred to deductive reasoning. The various theorems are arranged in groups, one outstanding theorem in each group being regarded as fundamental. The proof of each derived theorem is not here written out in full, but the method of proof is indicated by means of exercises on the properties of the figure, which is repeated with the same lettering when the full proof is set out in Part III. Thus no restriction is placed upon the use of the textbook preferred by the teacher, though the authors express the hope that this arrangement will prevent undue stress on the proofs of theorems in the early stages.

There are abundant exercises, showing variety and interest, carefully graduated and arranged in sets of calculations, constructions and riders : easy ones for the many and harder ones for the few who are capable of more rapid progress.

The first seven chapters deal with Angles at a Point, Parallels, Angle-sum of Triangle and Polygon, Congruent Triangles, Isosceles Triangles, Constructions, Parallelograms and Mid-Point Theorems, together with introductory chapters on Solid Geometry, Loci, Scale Drawing and Similar Figures, followed by a carefully graded set of Revision Exercises.

The next eight chapters deal with Areas, the Theorem of Pythagoras and the Geometry of the Circle, followed by another carefully graded set of Revision Exercises.

Part II (chapters 16 to 26) fills in gaps deliberately left in Part I. The increased emphasis on deductive reasoning is now apparent. The subject matter includes a second chapter on Loci (calling attention to the double properties, inclusive and exclusive, of a locus) with a second, and illuminating, chapter on Solid Geometry. Other chapters deal with Theorems about Proportion and Similar Triangles, Applications of Similar Figures, Use of Algebraic Identities in Geometry, Extensions of Pythagoras' Theorem, Inequalities, Converse Theorems, and, to attract the stronger pupils, a brief Introduction to More Advanced Geometry, a few Miscellaneous Theorems, and a short chapter, entitled Revision of Theorems, in which the authors take the interested pupil into their confidence and lead him to trace out, by working forwards or backwards, the logical structure by which the many derived theorems are seen to depend upon certain definite assumptions.

Part III gives the assumptions, the chain of derived theorems neatly and clearly proved in a concise form, and the usual standard constructions. It contains no solid geometry. The use of single small letters to denote angles not only reduces the space occupied by the proofs but also makes them easier for the pupil to follow. The treatment of Parallels is based upon Playfair's Axiom. Of the 58 theorems, six only are treated as axioms. This section is entirely bookwork, the numerous exercises being all in Parts I and II.

On page 94 a footnote, " See Notes and Answers ", suggests that the present edition of this separate publication is an advance copy, not yet complete. On page 7 of this edition insert " Revision Exercises A " between lines 16 and 17 ; and on page 12 insert " Revision Exercises B " between lines 9 and 10.

A conspicious feature of the work is the frequency of three-dimensional exercises. Such questions occur even before the preliminary chapter 4 on Solid Geometry, and thereafter they appear in separate groups throughout Parts I and II. As three dimensions are included wherever possible, should not the words " in one plane " be inserted wherever necessary?—for instance, in many of the early theorems, such as theorem 2, which reads, " If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line ".

The early " Introduction to Similar Figures " following a chapter on Scale

Drawing has already been mentioned. The pupil is expected to perceive intuitively that if two triangles XYZ and RST are equiangular, then corresponding sides are respectively proportional, and to check his intuition by measurement. Perhaps a better check would be to assume that the smaller triangle XYZ could be enlarged without change of shape until XY becomes equal to RS and then by the A.S.A. congruence XYZ becomes $\equiv RST$. The conditions for similarity of two triangles, corresponding to those for congruence are treated as fundamental, and groups of exercises using similarity appear at frequent intervals in Parts I and II.

The formal treatment of Similar Triangles in Part III is based on the proportional division of two sides RS and RT of a triangle by a parallel XY to the base ST. Here, perhaps, the authors have missed an opportunity. If the parallelogram $YXSW$ is completed, we have

$$XY/ST = SW/ST = RY/RT = RX/RS,$$

showing that *A line parallel to the base of a triangle cuts off a triangle similar to the original.* The three theorems on the similarity of two triangles RST and XYZ may then be deduced from the three analogous theorems on congruence by means of one and the same construction: along XY take $XH = RS$ and let HK, parallel to YZ, meet XZ at K. Then, in every case, XHK is similar to XYZ and congruent to RST.

On pages 278 and 279 the authors consciously give incomplete proofs of theorems dealing with angle conditions for four concyclic points. It seems a pity to say that a certain circle must either (i) or (ii) or (iii) and then point out in a footnote that there are really *five* alternatives, not *three*. Why not proceed somewhat as follows?

(i) If RS subtends equal acute angles α at P and Q on the same side of RS, the circumcentre both of PRS and of QRS is the vertex O of an isosceles triangle ORS on the same side of RS as P and Q and having a vertical angle 2α—conditions which uniquely determine the position of O.

(ii) If RS subtends equal obtuse angles α at P and Q on the same side of RS, the circumcentre both of PRS and of QRS is the vertex O of an isosceles triangle ORS on the other side of RS and having a vertical angle $2\pi - 2\alpha$—conditions which again uniquely determine the position of O.

(iii) If RS subtends an acute angle α at P on one side of RS and its supplement $(\pi - \alpha)$ at Q on the other side of RS, the circumcentre of QRS is the vertex O of an isosceles triangle ORS on the same side of RS as P and having a vertical angle $2\pi - 2(\pi - \alpha) = 2\alpha$. Thus again PRS and QRS have the same circumcircle.

In *A Second Report on the Teaching of Geometry in Schools*, to which reference has already been made, there is a suggestion that at some future time "it is not unlikely that the strictly geometrical part of the course would be lightened by the introduction of trigonometry in somewhat the same way as it has already been lightened by the use of algebra". The authors of the textbook under review are evidently in full agreement with the above report that the time for such a fusion is not yet. There are occasional references to trigonometry as to a separate subject, and indeed the *cosine* of an acute angle is defined on page 55.

I have noticed only two trifling misprints:

p. 167. 2nd line from bottom, for RXT read RTX.
p. 248. 2nd line from bottom, for (ii) read (iii).

On page 191, the statement that a solid of uniform cross-section is a prism "if the cross-section is a figure composed of straight lines" does not seem satisfactory, though the later reference to "the generators" may correct a wrong impression.

In dealing with areas of triangles and parallelograms, the words " on the same base and between the same parallels " might well be changed to " on the same base and between that base and a parallel to it ". In this book, however, the addition in brackets of " or, of the same height " prevents misunderstanding.

In Part III there is no formal proof of propositions converse to those which deal with the rectangle properties of a circle. These converses are however quoted on page 138 and set as exercises on page 217.

As a textbook suitable for the present state of development in school teaching of Elementary Geometry this book is outstanding. Not many boys who have the privilege of using it will find the subject dull. Nor will the treatment hamper clever boys, but will, if properly handled, stimulate their progress; and so much help is afforded both to teachers and to taught as to render it difficult for any user to mishandle the book. W. J. D.

Rational Geometry for Schools. I, II, III. By S. H. GLENISTER. Pp. 64; 62; 64. 1s. 6d. each part. 1938. (Harrap)

There is much to praise in this little book on practical geometry; the exercises are varied and interesting, they are carefully graded in order of difficulty, and the author has been at evident pains to bring out both the artistic and utilitarian sides of the subject. He has been greatly assisted by the publisher; the format is good, diagrams are clear and neat and there is an attractive air of spaciousness throughout the book.

No preface is given, but the work seems intended as a self-contained course for use in elementary and senior schools. It is arranged in the form of lessons, the main part of each lesson being to copy the drawing of some familiar object; but this is varied by scale drawing, geometrical constructions and plan and elevation exercises. There is, too, a progressive course of pattern drawing which includes architectural designs.

Two comments are offered, but more in the spirit of suggestion than criticism. There seems to be overmuch copying of copies; where possible, the object itself, brick, paperweight, or whatever it is, should be at hand and its actual dimensions utilised for the exercise. There are, indeed, examples where this principle has been adopted, but one would like to see it in evidence throughout. Again, in connection with the lessons on development of surfaces, it might be made quite clear that the exercises are not to finish with the mere drawing; they must always go on to the actual construction of the model. But this is perhaps regarded as self-evident by the author and, in any case, the matter can easily be put right. The course may be assessed as well designed to promote neatness and accuracy, to encourage the growth of a spatial sense in the pupils, and to serve as a valuable introduction to more formal work in architectural and mechanical drawing. H. L.

Progressive Geometry, First Stage. By H. W. BRISTER and W. G. E. DUKE. Pp. xii, 178. 3s. 1938. (Oliver and Boyd, Edinburgh)

This enterprising course of non-deductive geometry is based on the thesis that the aim at first should be to develop a spatial sense rather than to check up theorems. The approach is made by way of simple solids, cube, cylinder, pyramid and so forth, certainly the correct psychological procedure, though we should prefer less academic examples such as the rabbit-hutch and garage for the very beginning. From the construction of these models the pupil is led, by easy stages, to acquire a considerable amount of space lore, including even Euler's Theorem on polyhedra.

Proceeding to the plane, the course follows broadly the usual sequence of theoretical geometry, parallels, congruent triangles, parallelograms and areas.

Many of the properties are, indeed, obtained deductively, and the boy is progressively trained to set out his ideas in a reasoned and orderly, though not formal, manner. Here we suggest that, since the orthodox setting-out is ultimately essential, an occasional excursus in this direction would be valuable as preparation. The discussion of congruency ought, we feel, to come from the problem of constructing a model of definite size; the question arises naturally in this connection and is thus freed from any appearance of unreality. Areas, too, can be related, without much forcing, to the amount of cardboard required for a model.

Sections of solids are early dealt with and should present no particular difficulty if wire figures are used in conjunction with the closed models; we consider, however, that such a topic as the angle between skew lines is best relegated to the far end of the course or even omitted altogether from this stage. On the other hand, we should prefer the regular polygons to be brought in as soon as possible and with them pattern drawing, tessellated pavements, for example, and the construction of regular polyhedra from their nets; they are all quite easy, bring interest to ruler and compass work, and appeal to the child's aesthetic sense.

Mention must be made of an instructive section on loci, wherein such resources of modern boyhood as meccano rods are utilised in the attractive art of curve tracing.

It will be evident that the course has been planned with care and insight; it is, moreover, the outcome of actual work in the class-room. And yet we cannot help feeling a little doubtful about the end achieved. If this course is to be followed by another, on sternly deductive lines, covering much of the work here dealt with, we regard it as an excellent preparation and have no further misgivings. If, however, it is meant to be self-contained, it seems to us inadequate. This is not to disparage the authors, but is merely our view on the general position. After all, the value of deductive work lies not only in the training it gives in logical arrangement, but even more in the fact that it builds up an ordered edifice in the boy's mind, consolidating at every stage the knowledge acquired. No informal course can, in our opinion, serve the same purpose. There is a certain element of wordiness, too, probably inseparable from the general plan. Why should it take two pages of close print, for example, to *suggest* that an isosceles triangle has two equal angles? Surely the bare mention of a pair of steps or a picture cord would do as well. In their effort to avoid formality the authors lose crispness, and we are sure that no child of twelve, say, can keep up the concentration required to follow some of the discussions. The wording of examples is often obscure and the marking of figures, *e.g.* those on p. 88, occasionally falls short of the general level. All the same, the arrangement of the work is pleasant to the eye and much thought has been devoted, in the letterpress, to making the important points stand out boldly and clearly.

Taken altogether, the course possesses many commendable qualities, not the least of which is the evident enthusiasm of the authors. In the hands of a keen and sympathetic teacher it will certainly interest and stimulate the pupils, giving them that general sense of mental freedom which the subject so often fails to develop. The appreciative foreword by Sir Percy Nunn will be readily endorsed, even by those who still consider "Euclid" as essential to the armoury of every geometer. H. L.

Commercial Mathematics. By H. L. JONES. Pp. viii, 287. 4s.; without answers, 3s. 6d. 1938. (Arnold)

Some examining bodies have recently revised their syllabuses in commercial

arithmetic to include questions of a wider mathematical character. The author, who has based this book partly on his "Commercial Arithmetic", has written a parallel course to comply with the new syllabuses. The book consists of three parts, the first, which is short, consisting of a revision course of arithmetic of the standard of the middle forms of a secondary school. This revision course has 16 sets of exercises arranged according to subject, and is not accompanied by any explanation or comment.

Parts 2 and 3 are devoted partly to the usual course in Commercial Arithmetic and partly to a course in Algebra. With the limited space allotted to the latter, the author set himself a difficult task, and, considering the difficulties, has succeeded very well. This is due to the full emphasis he has put on the functional aspects of the subjects. This section would indeed do credit to any algebra intended specifically for a school course in the subject. In particular, the author is to be congratulated on his illustrations of logarithmic and semi-logarithmic graph papers and their uses.

With regard to the Commercial Arithmetic course, this has been well written, and the explanations are clear and easy to follow. The author's statement of the third, ninth and tenth rule is incorrect to the extent of £1 in £10,000.

There is another error which should be considered a serious one for a Commercial Arithmetic, although it is very common in even the most recent school arithmetics by the best authors. I refer to the author's treatment of Troy weight. He frequently mentions the Pound Troy in his examples. The Troy Pound does not exist except in the imaginations of authors of school arithmetic books—it has been dead for generations. The pennyweight too has been abolished, and the Troy Ounce is not divided into grains. The only way in which gold and silver is measured is by the Troy Ounce, which is sub-divided decimally as far as 0·001 oz. Any other system has been illegal for 60 years. With certain smaller and illiterate jewellers (and a large number of these exist), the names pennyweight and grain are still in use for weights marked 0.05 oz. and 0·002 oz. The former is equal to the defunct pennyweight, but the latter, of course, is not equal to the former grain. Nobody—not even an illiterate jeweller—uses the name Pound Troy, which is now in the same category as the Tower Pound and other defunct pounds. It should be noted that the grain is still used in Apothecaries' Weight. S. INMAN.

Mathematics for Technical Students, I. By A. GEARY, H. V. LOWRY, and H. A. HAYDEN. Pp. viii, 314. 4s. 1938. (Longmans, Green)

This very moderately-priced book is intended to provide the first year's work in a three years' course leading up to the national certificates in engineering, building and chemistry. The work is attractively presented and the exercises, many of which are drawn from practical applications, are both numerous and interesting. Sufficient revision is promised in succeeding parts to allow for variations in the syllabuses of different technical institutions. The series should prove extremely useful, so it may be worth while indicating in detail the ground covered by the present volume.

It is divided into five sections—Arithmetic (27 pp.), Algebra (119 pp.), Geometry (69 pp.), Mensuration (15 pp.), and Trigonometry (69 pp.). The first section deals with the multiplication and division of decimals (including contracted methods), factors, fractions, and ratio, proportion and percentage. The next section introduces the use of symbols, formulae and simple equations; directed numbers are briefly but adequately discussed; simultaneous equations are dealt with, and easy quadratic equations (by the use of factors); there are also chapters on graphs (including examples on the determination of straight-line laws), indices and logarithms, ratio, proportion and variation. The

geometry comprises the ideas of similarity and symmetry, the congruence of triangles, ruler and compass constructions, the properties of parallels, areas, Pythagoras' theorem, loci, and the angle properties of circles. In the beginning the ideas are based on intuition and drawing; proofs are introduced later, but teachers who do not wish or cannot afford to spare time for these will find that the exercises are designed so that the necessary information can be obtained by deduction from drawing. The mensuration deals with the area of irregular figures and the properties of simple solids (cylinder, pyramid, cone). The last section introduces the trigonometrical ratios of acute angles, the solution of right-angled triangles and simple examples on vectors; and finally extends, with the aid of graphical methods, the definition of sine and cosine to angles of any magnitude. This has been done early in the course as the authors think that engineering students are likely to meet with the idea of a periodic function in their other studies and so should become familiar with their graphs as soon as possible. Short historical notes are provided to add to the interest of the presentation of the subject.

The book is well planned and the pages have a pleasant appearance. One or two minor points may be worth mentioning. The statement on p. 146 "At the time of his death, about 550 B.C., Pythagoras was a young man of twenty" is liable to misunderstanding, as it is not obvious that "his" refers not to Pythagoras but to Thales, mentioned in the previous paragraph. In the construction on p. 210, the existence of a second tangent from P to the circle should have been indicated, especially as some of the later examples imply a knowledge of the properties of the figure formed by two tangents from a point to a circle. There appear to be misprints in Ex. 22 on p. 145 and the last line of p. 253, but, on the whole, the authors have done their work very well. W. H.

A Musical Slide-rule. By Lt. S. Lloyd. Pp. 25. 2s. 1938. (Oxford)

That section of a textbook on acoustics which deals with the musical scale and with temperament is often too mathematical for the student of music. With the aid of the musical slide-rule which he has designed, the author deals with these matters visually instead of mathematically, and does so with considerable success. The slide-rule (contained in a pocket at the back of the book) consists of two pieces of thick cardboard, showing on one side the intervals of the true scale, and on the other frequencies, and the intervals for mean-tone and equal temperament.

The main part of the book falls into four sections. In the first the author shows that most of the notes of the scale have to be varied slightly in pitch according to the chord of which they form part, so that the musical scale is a system, not of fixed notes, but of fixed intervals. The second section deals with frequencies and the method by which the intervals are marked on the slide-rule; the third shows that the slide-rule is graduated logarithmically, like its mathematical namesake; and the fourth section deals with temperament.

This is an interesting and useful piece of work, neatly planned and well produced. The actual slide-rule may be obtained separately for 1s. A. H. G. P.

Medieval Number Symbolism. By V. F. Hopper. Pp. xii, 241. 15s. 1938. (Columbia University Press; Oxford University Press)

The author states in his preface that the purpose of this study is to reveal how deeply rooted in medieval thought was the consciousness of numbers, not as mathematical tools, . . . but as fundamental realities, alive with memories and eloquent with meaning.

The subject is discussed under the following headings, the early chapters

describing the sources, the later ones the meaning and influence of the Christian, which was the dominant number symbolism in Medieval Europe: I. Elementary number symbolism; II. The astrological numbers; III. Pythagorean number theory; IV. The Gnostics; V. The early Christian writers; VI. Medieval number philosophy; VII. The beauty of order—Dante. There is an appendix on number symbols of northern paganism.

Professor Hopper reminds us that the Old and New Testament alike, though repeating the significant numbers of Babylonian astrology, are almost devoid of deliberate number symbolism, and the early Christian fathers were men without special knowledge of numbers. He then traces the growth of number symbolism from the admission into the church of scholars who looked for more than a literal meaning in the numerical passages of the Scriptures, and set themselves among other tasks that of explaining a universe in which spiritual and material truth were coordinated in a single finite plan. The solution was sought in numbers, and the existing number philosophies were pressed into service and elaborated. Several of the arguments on numbers are quoted. There are also illustrations of symbolic numbers occurring in theological and general literature, cathedral architecture and the offices of the church. They show how the earnest work of these scholars impressed upon others that numbers were real and vivid with meaning.

There is an important chapter on Dante and the influence of number symbolism on his thought, a field of research in which the reviewer is a stranger.

Much work has been put into this book. It is full of information supported by more than a thousand footnote references. The index lists the meanings of the numbers in each philosophy. There are twenty pages of bibliography.

The book touches mathematics at a few points only. Nevertheless, anyone whose work is with numbers will find in it much to interest him. F. A. Y.

Magische Quadrate und Magische Parkette. Pp. 78. RM. 2. 1937. Scientia Delectans, Heft 2.

Der Keplersche Körper und Andere Bauspiele. Pp. 65. RM. 2. 1938. Scientia Delectans, Heft 3. By Dr. Gerhard Kowalewski. (K. F. Koehlers Antiquarium, Leipzig)

The first of these monographs claims to be a new contribution to the ancient study of magic squares. Magic squares are of Chinese origin, and it is known that some were constructed in India before the Christian era. They were introduced into Europe by Moschopulus in the fifteenth century. The development of the theory of magic squares by the French mathematicians roughly occupied the period from 1600-1800. Except for occasional later references, magic squares seem now to have been relegated almost entirely to books on mathematical recreations. It is interesting, therefore, to note that the author has found a new contribution to make to the study of these fascinating squares.

The author considers two classes of four-rowed magic squares, which he calls Dürer or Sun Squares according as they satisfy certain relations termed "cross-conditions". If the rows of a square array of sixteen numbers are denoted by $a_1, a_2, a_3, a_4 : b_1, \ldots, b_4 : c_1, \ldots, c_4 : d_1, \ldots, d_4$, two *crosses* can be formed, one by joining the pairs of points a_2, d_2; c_1, b_4 and the other by joining the pairs a_3, d_3; b_1, c_4. The Dürer squares satisfy the two "cross-conditions" $a_2+d_2=b_4+c_1$, $a_3+d_3=b_1+c_4$. The two crosses in this case have *equal weight*. For the Sun squares, on the other hand, the difference between the weights of the two crosses is as great as possible.

By means of these "cross-conditions" the author discusses the construction of magic squares of both types and also extends his methods to what are usually termed pan-magic squares.

The second volume is concerned with one of Kepler's solids. These polyhedra, which can be formed from the five platonic regular solids, were studied by Kepler in great detail in 1596, but many present-day mathematicians are probably unfamiliar with them. The book is concerned, in particular, with that one of Kepler's solids whose thirty faces are rhombuses. The solids are here constructed from two groups of ten vari-coloured blocks built up and arranged according to the different colours of their faces, on the domino principle, in the same way as MacMahon cubes. The method of construction is described in detail and with great clarity in the booklet.

The subject matter of this monograph is dealt with somewhat less seriously than that of the preceding, and readers may well be both fascinated as well as amused by trying to play the author's " colonisation game " with the thirty little men whose coats and trousers are of different colours ! Lest the reader should be tempted to regard this game as a mere bit of light amusement with which to fill up a weary hour, he should be warned that the author expresses the hope that his game will arouse interest on account of its difficulty ; and anyone who hopes to play it without first acquiring the necessary technique may well feel aggrieved that the author has called this complicated but ingenious device a " game ".

The little booklets in this new series are clearly printed and attractively produced. They deal with subjects rather outside the field which is very familiar to present-day professional mathematicians in this country, and for this reason they may be found all the more refreshing and stimulating to their readers. E. G. P.

Elementary Mathematical Statistics. By W. D. BATEN. Pp. x, 338. 15s. 1938. (John Wiley and Sons, New York ; Chapman and Hall)

This book is by the Associate Professor of Mathematics, and Research Associate in Statistics, of the Michigan Agricultural Experiment Station, Michigan State College, and is intended as an elementary introduction requiring no calculus. The treatment is therefore in the main simple, three chapters (on charts, index numbers, and time series) being interpolated in the main course through averages, dispersion, etc., probability and normal equations to correlation (coefficient and ratio, multiple and partial) sampling and variance. The author expresses the hope that the book will lead the student to further study : it is a pity that the references are so scarce, both to original work (Irving Fisher's book on Index Numbers is referred to with no date nor publisher, and R. A. Fisher simply mentioned in passing) and to the source of the statistical tables that he gives (*e.g.* the index number tables of 8·2, 8·4, 8·7, and 8·8 appear to be associated yet the first is in dollars and the last speaks of pennies) and that there is no bibliography to guide to further study.

The book is " built " in the U.S.A., and we find the printing unsatisfactory to our eyes. Indices are put (*e.g.* p. 117) on the same level as the numerator of their fractional bases, though a different sized type is used, while for subscripts different sized types seem to be used for digits and for letters, the latter being sometimes very difficult to read. Misprints include v (for v_i, p. 54), radiates (for radiating, p. 59), traget (for target, p. 118), v_2 (for v_3, p. 120), derivations (for deviations, p. 156), C_1 (for c_1, p. 190), multiple (for multiple, p. 269), and M_z (for $\sigma_{M_{z_0}}$ p. 277). We do not like the author's definition of probability in terms of " equally likely " events (p. 113), his use of the word " percentages " for " fractions " (pp. 88, 121), of " expected number " (for " most probable number ", p. 126), nor of " prices " in the sense of money values as distinct from a rate (*e.g.* p. 132). Latin Square is not defined, nor is there any explanation of the 2·6 factor that he uses on p. 220 as indicative

of significance. Question 4 of p. 129 seems out of its place before sampling has been dealt with, the notation 5^+ (p. 61) and 5^- (p. 63) is not explained, whilst the area for $t = -4{\cdot}09$ is not found in Table 1 as stated on p. 88. In the absence of any measure of the standard error of skewness, the statement on p. 126 that 0·067 is not far from the skewness of a normal distribution is unjustified. On the other hand, the writer's distinction between "open" and "closed" limits is useful, though he does not always (*e.g.* pp. 81, 101, 107) follow his own advice.

The appendices include five-figure tables for area and ordinate of the normal curve, Snedecor's table for levels of significance for n_1, and n_2 (though on its first introduction, p. 253, there is no indication that the two entries give the 5% and 1% points, nor does he make it clear whether the t of this table has anything to do with the deviate t of Tables I and II) tables of n^2, $\sqrt{n}$, $1/n$, $\log_{10} n$, and $\log_e n$ (though e does not appear to be mentioned elsewhere in the book and none of his methods require $\log_e$). The book closes with answers to the questions to the numerous examples that the author has included: the student would probably like something more than a mere "Yes" as the answer to the owls, brown and white mice question of p. 271. And the point (p. 113) whether problems can be solved by formulas for permutations, combinations, or both, is not one that is determined solely by experience, as the author suggests.

FRANK SANDON.

Introduction to Bessel Functions. By F. BOWMAN. Pp. x, 135. 10s. 6d. 1938. (Longmans, Green)

Prior to 1934 a knowledge of the elements of Bessel functions could only be obtained from well-known treatises. It was not easy for the student to wade through a mass of information and select what he ought to study. Now, however, there are several works which make the beginner's task less arduous. The present volume is the second of this kind to be printed in English. The text deals mainly with functions of zero order. In chap. 1 the author starts by defining $J_0(x)$ as a power series, and finds the differential equation of which it is one solution. The other solution is also found. Then follow simple integrals, the behaviour of $J_0(x)$, $Y_0(x)$ when x is large; roots; Fourier-Bessel and Dini expansions of zero order. In chap. 2 there are applications to the symmetrical modes of a freely vibrating membrane, etc. The function $I_0(x)$ is applied in chap. 3 to skin effect in a straight circular wire carrying alternating current, and the ber, bei functions are introduced. Various exponential-Bessel integrals of zero order are given. Next come asymptotic expansions of the Hankel and Bessel functions of zero order, followed by a chapter on functions of any real order. The book closes with applications to planetary motion, the elastica, and the general case of a freely vibrating membrane. In each of the seven chapters there are numerous examples to be worked out by the reader.

From a mathematical viewpoint the treatment is simple, concise, and as rigorous as the type of text requires. It will undoubtedly be a valuable introduction to the subject of Bessel functions for the student of mathematics. The text is not well suited for engineers—despite the claim on the loose cover!—because the general atmosphere of the applications is too far removed from that of their practical activities.

There are several points which call for comment: Some of the symbols differ from those in common use, *e.g.* x instead of I for current. According to long established usage the expression "equivalent" resistance should be "effective" resistance. The applications proceed from general to particular, whereas the reverse order is preferable for students. In dealing with skin

effect, the polar form of the functions would have been simpler and easier to apply than the cartesian form. With asymptotic expansions it often happens that the terms decrease at first and then commence to increase, so that the greatest accuracy is obtained by stopping at the smallest term. This should have been mentioned, as it is important in numerical work. By giving contour integrals in chap. 5 without any introduction to the subject, the author assumes that his readers are already familiar with complex integration. The pages devoted to this subject will be beyond engineers and many physicists. Inclusion of the exponential method of expressing powers of i would have been welcome. The definition of ber x and bei x from $I_0\left\{\left(\frac{1+i}{\sqrt{2}}\right)x\right\}$ is unusual. The standard way is $J_0(i^{\frac{3}{2}}x) = \text{ber}\, x + i\,\text{bei}\, x$. These comments are to be regarded more in the nature of suggestions than criticisms. N. W. McLachlan.

Theory of equations. By J. M. Thomas. Pp. x, 211. 12s. 1938. (McGraw-Hill)

This is a useful textbook on a subject which is a most important auxiliary in many investigations, both practical and theoretical. It deals with the elementary part of the theory, stopping short of the Galois theory, but the treatment is on modern lines, with due attention to careful statements concerning the number fields in which results are true. Chapters on permutations and determinants make a precise treatment of systems of linear equations easy; another excellent chapter is that on "Constructibility", which discusses ruler and compass constructions and then deals with the "famous problems" of the trisection, the duplication of the cube and the construction of regular polygons. Most of the bookwork is expounded with a crispness that makes for pleasant reading; an exception is the chapter on "Simultaneous systems", the first part of which I found difficult to follow and the second part, on the location and computation of imaginary roots, unconvincing. Two other points may be mentioned: Descartes' rule of signs is deduced from Budan's theorem, but it might have been better to give a more direct proof in the text, leaving the deduction from Budan's theorem as an example; in Horner's method, the roots are not successively multiplied by 10, an omission which, in my opinion, is likely to increase the possibility of arithmetical errors.

If we take into account the excellent printing, good paper and strong binding, twelve shillings is not an unreasonable price for a self-contained textbook of this type. But it should be added that there is really very little which cannot be found in two or three chapters in modern algebras, for example, Durell and Robson, *Advanced Algebra*, so that for the average teacher, obliged to restrict his purchases severely, this is likely to remain a luxury volume, especially as numerical computations are rather inadequately treated. A school library might find it useful. T. A. A. B.

An Introduction to Co-ordinate Geometry. By W. S. Catto and F. J. H. Williams. Pp. 212. With answers, 3s. 6d. Without answers, 3s. 0d. 1939. (Harrap)

A Scottish School Geometry, part IV. By W. G. Thomson. Pp. 86. 2s. 0d. 1938. (Oliver and Boyd)

In the last number of the *Gazette* was reviewed one little book on Analytical Geometry especially written for higher grade candidates in the Scottish Leaving Certificate, and here are two more—one by two members of the staff of George Watson's College, Edinburgh, and the other by a member of the staff of Robert Gordon's College, Aberdeen. The book from the capital is naturally taken first, and incidentally it is by far the better book of the two.

One of the most attractive features of the first-named production is the large number of examples that are worked in full in the text. These will be a great boon to those pupils who will be working at the subject under occasional supervision. The book starts well in giving on the first five pages a summary of the important facts that have to be memorised, although the value of this list would be enhanced if each formula was followed by the number of the page on which its proof appears. Personally I do not like the idea of memorising the formula for the area of a triangle the coordinates of whose vertices are given, unless it is in the form of a determinant, but prefer beginners to work each example *ab initio*. The coordinates of the middle point of the straight line joining (x_1, y_1) and (x_2, y_2) are found before those of the point dividing the same line in the ratio of m to n. This is not to be deprecated, but surely it is an omission not to show that the first may be deduced from the second. When loci are introduced, Messrs. Catto and Williams emphasise that the equation of a locus is true for all points whose coordinates satisfy that equation, and is not true for any other points. In this they show that they are experienced teachers, for it is one of the facts of the subject that beginners find difficulty in appreciating. A straight line is defined as the locus of a point which moves so that the gradient of the path traced out is constant, and from this definition is proved—very neatly indeed—that the general equation of the first degree represents a straight line. In dealing with the bisectors of the angles between two intersecting straight lines, the authors give a diagram which shows the differences in sign of the perpendiculars drawn from a point on a bisector according to the position of the point. This is then followed by another diagram, which is merely a part of the first. I would suggest that the first diagram be omitted altogether, and the information given in it be given in two diagrams of which the second is one. The alternative methods for finding the equation of a chord of a circle seems unnecessary and not at all convincing to a boy or girl, for they commence—" Consider the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=x^2+y^2-r^2,$$

or,
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=x^2+y^2+2gx+2fy+c\text{ "},$$

a type of beginning which always appears to a pupil to be a little unfair. The section on the common chord of two circles seems to need some alteration. The equation of the common chord is found by solving the equations of the circles simultaneously for x and y, and then two pages later the following example is worked—" Consider the circles

$$x^2+y^2=9, \quad \text{and} \quad x^2+y^2-16x-12y+75=0,$$

the common chord of which is found to be $4x+3y=21$. On eliminating y this gives $25x^2-168x+360=0$, a quadratic equation whose roots are not real. Therefore the circles have no real point in common, and thus have *no* common chord." Now consider a boy or girl of fifteen or sixteen being confronted with two *real* circles, which apparently have a *real* common chord, which they are then told does not exist! If they seek to find an exit from this tangle by actually drawing the circles they will find that there is certainly no common chord. Therefore what is this line which is a common chord that is not a common chord! Consequently, it would be better to commence the section by defining the Radical Axis of two circles, find its equation, and show that it is the common chord when the two circles intersect in real points. It is too much to expect pupils at this stage to appreciate the subtlety of a real straight line drawn through two imaginary points. This is the most serious blemish in an otherwise excellent production which is a worthy successor to the Trigonometry by the same authors published in 1936.

The outstanding feature of Mr. Thomson's book is the large number of footnotes. In the preliminary section on coordinates there are five footnotes in seven pages; in the section on the straight line there are twenty-six footnotes in forty pages; and in the section on the circle there are eleven pages and only one footnote. These footnotes spoil the book. To a boy or girl if a fact is only worthy of a footnote then it is not worth bothering about! Another displeasing feature is the style in which the theorems are given, for there does not seem to be a clear-cut statement from which is deduced another clear-cut statement, and so on; in fact the impression given is that there is too much talking and not enough is said. The perpendicular distance of a point from a given straight line is usually found fairly difficult, particularly when an explanation is required of the double sign in the formula $\pm(ax_1+by_1+c)/\sqrt{(a^2+b^2)}$. Mr. Thomson finds the distance of (x_1, y_1) from $x\cos\alpha+y\sin\alpha-p=0$, and merely adds as a corollary: "The directed distance of a point (x_1, y_1) from a straight line $ax+by+c=0$, where c is negative, is $+(ax_1+by_1+c)/\sqrt{(a^2+b^2)}$". I have always found that this requires a great deal of explanation before a pupil manages to grasp it. Also he makes the same mistake as Messrs. Catto and Williams over the common chord of two circles. However, Mr. Thomson gives 472 exercises to be worked, which seems remarkable value for the money. A. I.

1. Elementary Plane Trigonometry. By H. E. Piggott. Pp. xii, 243. 7s. 6d. 3rd (1925) edition, reprinted 1938. (Constable)

2. A Concise Trigonometry. By S. H. J. Cox. Pp. vii, 183, xiv. 3s. 0d. 1938. (Rivingtons)

Both Beethoven and Sydney Smith wrote variations on "The Blue Bells of Scotland", but whereas Beethoven exercises and enlivens the original theme, the only change we get out of Sydney Smith is in trivial and "exercisey" accompaniments. Of course Beethoven cared chiefly about the musical value, while Sydney Smith doubtless had uppermost in his mind the Victorian drawing-rooms in which his works were so frequently "rendered".

These two books on Trigonometry are written from very different standpoints. Mr. Cox "has tried to provide, in the shortest space practicable, a course suited to the needs of candidates for the O. and C. Joint Board (Additional Mathematics) and for London Matriculation (Mathematics More Advanced). While the book is intended primarily as a teaching manual, the needs of those who must perforce teach themselves have been specially catered for."

Mr. Piggott, on the other hand, intends the text of his book to be read, "consecutively and completely". The improvement in oral teaching, he explains, though arousing a livelier interest in mathematical work, has caused the boys and girls of the present day to lose the habit of handling textbooks as sources of information, or of being able to follow, unaided, a line of mathematical reasoning. In order to make his book easily readable he has "above all tried to be coherent and logical; he has endeavoured to make the subject interesting by showing how, at every stage in the development, the information and methods may be applied practically—to problems occurring in connection with Surveying, Navigation, Engineering and Physics—and the reader is invited to cooperate by completing calculations left unfinished in the text".

Mr. Cox's conciseness is obtained chiefly at the expense of references to practical affairs, and he has produced a neat bundle of the dry bones of the subject, well stuffed with a surprisingly large number of neatly arranged exercises. There is not a graph to be found, and the three-dimensional work is almost non-existent. Coordinates are not used to define the ratios, and

Cauchy's geometrical proof of the addition formula is given, so that conciseness has been sacrificed at times in order to avoid the simplest algebraical geometry. Two pages devoted to "Dip of the Horizon" seem unnecessary. The treatment of the inequality $\sin\theta < \theta < \tan\theta$ is curious. The author first gives what he calls a "strict proof" based on the "obvious" relation between the areas of the two triangles and the sector, and adds that the student may satisfy himself of the truth of the inequality intuitively by examining a figure, which makes it clear that chord, arc and tangent are in ascending order of magnitude! The language is often irritating: "The line OP is to be regarded as plus in any position", "an angle $\alpha°$, where α is any finite positive magnitude". There are many redeeming features, however. The printing is good, and Mr. Durell's well-known tables are incorporated. The worked examples are numerous, there are valuable general hints, and notes on dimensions, symmetry, and irreversible operations. The absence of graphs, latitude and longitude and other simple three-dimensional work invalidates the author's claim to suit the O. and C. Joint Board Additional Mathematics.

Mr. Piggott's book is decidedly readable. Each stage of the work is made to grow from some practical need. The treatment of projections is based on simple problems in navigation, and gyro-compass bearings are used to introduce the general angle. Ideas of wider application, such as vectors, orthogonal projection, polar coordinates, simple harmonic vibration, and damped vibrations all fall naturally into place, and are not simply side-tracks. The boy who has worked through this book will not only have nothing to unlearn, but will have acquired a very sure foundation for more advanced work. There is an air of reality and reasonableness throughout, and if the author's position at the R.N.C., Dartmouth, is responsible for the slightly nautical flavour, surely it is as natural to talk about ships in elementary trigonometry as about shops in elementary arithmetic.

It is surprising to find the formula for $\sin\frac{1}{2}A$ preferred to that for $\tan\frac{1}{2}A$ in the solution of triangles with three sides given. The use of cologarithms (converted mentally from ordinary logarithm tables) is advocated. No review of any earlier edition has appeared in the *M. G.*, and for the benefit of those familiar with the first (1919) edition it may be said that the new version gives the ordinary "coordinate" definitions of the ratios for angles measured counter-clockwise from the x-axis, as well as for angles measured clockwise from "north", and an appendix dealing with the haversine has been displaced by an extra set of exercises on identities and transformations.

Mr. Piggott has been less fortunate in his choice of publisher than Mr. Cox. His only slightly larger book costs two and a half times as much, contains no tables, and the examples are printed in rather small type. Perhaps these defects can be remedied in a revised edition, and so enable a valuable work to reach a wider public. Mr. Piggott does all that he sets out to do, and more. His book in fact suits in content and in spirit the O. and C. Joint Board Additional Mathematics: in concentrating on teaching his subject as something worth knowing he is more likely to "get you through the exam." than any *ad hoc* crammer.

A. P. R.

CORRIGENDA.

October, 1938, p. 368, l. 4 below Fig. 5. Read "DAB less K is greater than EBA; that is, ADE is greater than ABE".

December, 1938, Note 1343, pp. 506-7. In the determinant (i), last column, third row, for "1" read "0". In the statement (iii), after "$X_1+X_2=\xi_1$" insert "$X_1-X_2=\xi_2$". Note 1344, p. 508. Professors Piaggio and Synge have pointed out that the proof is invalid, from confusion between dq_r, standing for $\dot{q}_r dt$, and δq_r.

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